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# A spectral inequality for degenerated operators and applications

Rémi Buffe\*, Kim Dang Phung†

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**Abstract .-** In this paper we establish a Lebeau-Robbiano spectral inequality for a degenerated one dimensional elliptic operator and show how it can be used to impulse control and finite time stabilization for a degenerated parabolic equation.

**Résumé .-** Dans cet article, on s'intéresse à l'inégalité spectrale de type Lebeau-Robbiano sur la somme de fonctions propres pour une famille d'opérateurs dégénérés. Les applications sont données en théorie du contrôle comme le contrôle impulsif et la stabilisation en temps fini.

## 1 Introduction and main results

The purpose of this article is to prove spectral properties for a family of degenerate operators acting on the interval  $(0, 1)$ . We shall consider linear operators  $\mathcal{P}$  in  $L^2(0, 1)$ , defined by

$$\begin{cases} \mathcal{P} = -\frac{d}{dx} \left( x^\alpha \frac{d}{dx} \right) , & \text{with } \alpha \in (0, 2) , \\ D(\mathcal{P}) = \{ \vartheta \in H_\alpha^1(0, 1) ; \mathcal{P}\vartheta \in L^2(0, 1) \text{ and } \text{BC}_\alpha(\vartheta) = 0 \} , \end{cases}$$

where

$$H_\alpha^1(0, 1) := \left\{ \vartheta \in L^2(0, 1) ; \vartheta \text{ is absolutely continuous in } (0, 1), \int_0^1 x^\alpha |\vartheta'|^2 < \infty, \vartheta(1) = 0 \right\} ,$$

and

$$\text{BC}_\alpha(\vartheta) = \begin{cases} \vartheta|_{x=0} , & \text{for } \alpha \in [0, 1) , \\ (x^\alpha \vartheta')|_{x=0} , & \text{for } \alpha \in [1, 2) . \end{cases}$$

We remind that  $\mathcal{P}$  is a closed self-adjoint positive densely defined operator, with compact resolvent. As a consequence, the following spectral decomposition holds: There exists a countable family of eigenfunctions  $\Phi_j$  associated with eigenvalues  $\lambda_j$  such that

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- $\{\Phi_j\}_{j \geq 1}$  forms an Hilbert basis of  $L^2(0, 1)$
- $\mathcal{P}\Phi_j = \lambda_j \Phi_j$
- $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow +\infty$ .

An explicit expression of the eigenvalues is given in [Gu] for the weakly degenerate case  $\alpha \in (0, 1)$ , and in [Mo] for the strongly degenerate case  $\alpha \in [1, 2)$ , and depends on the Bessel function of first kind (see [MM]). Also, we have the following asymptotic formula:  $\lambda_k \sim C(\alpha) k^2$  as  $k \rightarrow \infty$ .

We are interested on the spectral inequality for the sum of eigenfunctions. Our main result is as follows.

**Theorem 1.1** .- *Let  $\omega$  be an open and nonempty subset of  $(0, 1)$ . There exist constants  $C > 0$  and  $\sigma \in (0, 1)$  such that*

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq C e^{C\Lambda^\sigma} \int_{\omega} \left| \sum_{\lambda_j \leq \Lambda} a_j \Phi_j \right|^2 ,$$

for all  $\{a_j\} \in \mathbb{R}$  and  $\Lambda > 0$ . Further,

$$\sigma = \begin{cases} 3/4 , & \text{if } \alpha \in (0, 2) \setminus \{1\} , \\ 3/(2\gamma) \text{ for any } \gamma \in (0, 2) , & \text{if } \alpha = 1 . \end{cases}$$

Two different kinds of approach have been developed to obtain the spectral inequality for the sum of eigenfunctions: A first one is due to Lebeau and Robbiano [LR] and is based on a Carleman estimate for an elliptic operator, whereas a second one appears in a remark in [AEWZ] and is based on an observation estimate at one point in time for a parabolic equation. Note that in the standard setting of uniformly elliptic operator,  $\sigma = 1/2$  (see [L], [JL], [LZ], [Lu], [Mi], [LRR1], [LRLR]). In the present paper we will establish a new Carleman estimate for an associated degenerated elliptic operator. Because of the degeneracy of the coefficients of the operator  $\mathcal{P}$ , we make use of a new weight function in the design of the Carleman estimate. The subtle difference between the cases  $\alpha \in (0, 2) \setminus \{1\}$  and  $\alpha = 1$  is related to the existence of a Hardy type inequality for the  $H_\alpha^1$  norm. Indeed, for  $\alpha = 1$ , the desired Hardy inequality fails to hold.

Many applications to such spectral inequality have been developed, in particular in control theory (see [L], [LZ], [BN], [Le], [LRM], [BPS]). Let  $\omega$  be an open and nonempty subset of  $(0, 1)$  and denote  $1_\omega$  the characteristic function of a given subdomain  $\omega$ . We present the following two results.

**Theorem 1.2** .- *Let  $E \subset (0, T)$  be a measurable set of positive measure. For all  $y^0 \in L^2(0, 1)$ , there exists  $f \in L^2(\omega \times E)$  such that the solution  $y = y(x, t)$  of*

$$\begin{cases} \partial_t y - \partial_x (x^\alpha \partial_x y) = 1_{\omega \times E} f , & \text{in } (0, 1) \times (0, T) , \\ \text{BC}_\alpha(y) = 0 , & \text{on } (0, T) , \\ y|_{x=1} = 0 , & \text{on } (0, T) , \\ y|_{t=0} = y^0 , & \text{in } (0, 1) , \end{cases}$$

satisfies  $y(\cdot, T) = 0$ .

Theorem 1.3 .- *There is  $(t_m)_{m \in \mathbb{N}}$  a increasing sequence of positive real numbers converging to  $T > 0$  and  $(\mathcal{F}_m)_{m \in \mathbb{N}}$  a sequence of linear bounded operators from  $L^2(0, 1)$  into  $L^2(0, 1)$  such that for any  $z_0 \in L^2(0, 1)$ , the solution  $z = z(x, t)$  to*

$$\begin{cases} \partial_t z - \partial_x (x^\alpha \partial_x z) = \sum_{m \in \mathbb{N}} \delta_{t=(t_{m+1}+t_m)/2} \otimes 1_\omega \mathcal{F}_m(z|_{t=t_m}) , & \text{in } (0, 1) \times (0, T) , \\ \text{BC}_\alpha(z) = 0 , & \text{on } (0, T) , \\ z|_{x=1} = 0 , & \text{on } (0, T) , \\ z|_{t=0} = z_0 , & \text{in } (0, 1) , \end{cases}$$

satisfies  $\lim_{t \rightarrow T^-} \|z(\cdot, t)\|_{L^2(\Omega)} = 0$ .

Here  $\delta_{t=(t_{m+1}+t_m)/2}$  denotes the Dirac measure at  $t = (t_{m+1} + t_m)/2$ . Note that the above system equivalently reads

$$\begin{cases} \partial_t z - \partial_x (x^\alpha \partial_x z) = 0 , & \text{for } t \in \mathbb{R}^+ \setminus \bigcup_{m \geq 0} \left( \frac{t_m + t_{m+1}}{2} \right) , \\ z\left(\cdot, \frac{t_m + t_{m+1}}{2}\right) = z\left(\cdot, \left(\frac{t_m + t_{m+1}}{2}\right)_-\right) + 1_\omega \mathcal{F}_m(z(\cdot, t_m)) , & \text{for any integer } m \geq 0 , \\ \text{BC}_\alpha(z) = 0 , & \text{on } (0, T) , \\ z|_{x=1} = 0 , & \text{on } (0, T) , \\ z|_{t=0} = z_0 , & \text{in } (0, 1) . \end{cases}$$

Theorem 3.1 is new approach to steer the solution to zero at time  $T$  and can be seen as a finite time stabilization for the degenerated heat equation by impulse control. This can be compared with [CN]. The standard null-controllability problem is given when  $E = (0, T)$  and has been studied in [CMV]. It is now well-known that the null controllability for higher degeneracies ( $\alpha \geq 2$ ) fails to hold (see [CMV2] and the references therein). We also refer to [ABCF], where the null-controllability result has been extended to more general degeneracies at the boundary. When the control is located at the boundary where the degeneracy occurs, we refer to [Gu, CTY, MRR]. We finally refer to the recent book [CMV2] and the references therein for a full description of the field. Note that an estimation of the cost of controllability for small  $T > 0$ , as well as for  $\alpha \rightarrow 2^-$  has been recently obtained in [CMV3].

The outline of the paper is as follows. In Section 2, we present the key inequalities needed to prove Theorem 1.1 as Hardy inequality and Carleman inequality. Section 3 is devoted to obtaining the applications of the spectral inequality in control theory as observation estimates, impulse approximate controllability, null controllability on measurable set in time (see Theorem 3.4) and finite time stabilization (see Theorem 3.5). Theorem 1.2 and Theorem 1.3 are direct consequence of Theorem 3.4 and Theorem 3.5 respectively.

## 2 Key inequalities

This section is devoted to the statement of the key inequalities: Hardy inequality and Carleman inequality, that will enable us to prove Theorem 1.1. The proof of the Carleman inequality is given at the end of this section.

## 2.1 Hardy inequality and boundary conditions

The following Hardy inequality shall play a central role in what follows. The proof can be found in [CMV], [OK].

Lemma 2.1 .- *Let  $\vartheta$  be a locally absolutely continuous functions on  $(0,1)$  such that  $\int_0^1 x^\alpha |\vartheta'|^2 < \infty$ . Then we have*

$$\int_0^1 x^{\alpha-2} |\vartheta|^2 \leq \frac{4}{(2-\alpha)^2} \int_0^1 x^\alpha |\vartheta'|^2 ,$$

*if one of the following assumption holds:*

- i)  $\alpha \in (0,1)$  and  $\vartheta|_{x=0} = 0$  ,
- ii)  $\alpha \in (1,2)$  and  $\vartheta|_{x=1} = 0$  .

We also have the following lemma, that shall be useful when estimating the boundary terms arising from integration by parts in the strongly degenerate case  $\alpha \in [1,2)$ . The proof can be found in [CMV].

Lemma 2.2 .- *Let  $\alpha \in [1,2)$  and  $\vartheta \in H_\alpha^1(0,1)$ . Then  $(x|\vartheta|^2)|_{x=0} = 0$ .*

## 2.2 Global Carleman estimate near the degeneracy

In this section, we shall state the crucial tool, i.e. a global Carleman estimate near the degeneracy of an elliptic operator.

Introduce, for  $S_0 > s_0 > 0$ ,

$$Z = (-S_0, S_0) \times (0,1) , \quad Y = (-s_0, s_0) \times (0,1) .$$

First, we shall write

$$\mathcal{Q} := -\partial_s^2 + \mathcal{P} = -\partial_s^2 - \partial_x (x^\alpha \partial_x) , \tag{2.2.1}$$

here  $(s,x) \in Z$ . The weight function we choose is of the form

$$\varphi(\tau, s, x) = \tau \frac{x^{2-\alpha}}{2-\alpha} - \frac{\tau^{\gamma/3}}{\nu} s^2 , \tag{2.2.2}$$

where  $\tau, \nu > 0$  are two large parameters,

$$\begin{cases} \gamma = 2 , & \text{for } \alpha \in (0,2) \setminus \{1\} , \\ \gamma < 2 , & \text{for } \alpha = 1 , \end{cases} \tag{2.2.3}$$

and with  $\nu$  fixed sufficiently large. Note that this weight function is completely decoupled in the two directions, in particular with respect to the dependency in  $\tau$ . In the case  $\alpha = 1$ , the

Hardy inequality in Lemma 2.1 does not hold, and this is the reason of our subtle choice of weight (2.2.2). Next, we shall set

$$\mathcal{Q}_\varphi := e^\varphi \mathcal{Q} e^{-\varphi} .$$

Finally, we state a global estimate for functions of  $C^\infty((-S_0, S_0), D(\mathcal{P}))$ , with the proper weight function  $\varphi$  given by (2.2.2) to handle the degeneracy at  $x = 0$ .

**Theorem 2.1** .- *There exist  $\tau_0 > 0$ , and  $\nu_0 > 0$  such that for  $\gamma > 0$  defined in (2.2.3), there exists  $c > 0$  such that*

$$\tau^\gamma \|v\|_{L^2(Z)}^2 + \tau \int_Z x^\alpha |\partial_x v|^2 + \tau^3 \int_Z x^{2-\alpha} |v|^2 + \mathcal{B}(v) \leq c \|\mathcal{Q}_\varphi v\|_{L^2(Z)}^2 ,$$

for all  $\tau \geq \tau_0$ , and for all  $v \in C^\infty((-S_0, S_0), D(P))$ , where  $\mathcal{B}$  is a quadratic form satisfying

$$\begin{aligned} \frac{1}{2} \mathcal{B}(v) \geq & -\tau \int_{-S_0}^{S_0} |\partial_x v|_{x=1}^2 + 2 \frac{\tau^{\gamma/3}}{\nu} \int_0^1 [s |\partial_s v|^2]_{s=-S_0}^{s=S_0} + 2 \frac{\tau^{\gamma/3}}{\nu} \int_0^1 [v \partial_s v]_{s=-S_0}^{s=S_0} \\ & + 8 \frac{\tau^\gamma}{\nu^3} \int_0^1 [s^3 |v|^2]_{s=-S_0}^{s=S_0} - 2\tau \int_0^1 [x \partial_s v \partial_x v]_{s=-S_0}^{s=S_0} - \tau \int_0^1 [v \partial_s v]_{s=-S_0}^{s=S_0} \\ & - 2 \frac{\tau^{\gamma/3}}{\nu_0} \int_0^1 [x^\alpha s |\partial_x v|^2]_{s=-S_0}^{s=S_0} + 2 \frac{\tau^{2+\gamma/3}}{\nu_0} \int_0^1 [x^{2-\alpha} s |v|^2]_{s=-S_0}^{s=S_0} . \end{aligned}$$

Note that in the above Theorem 2.1, boundary conditions are prescribed through the membership in the domain of  $\mathcal{P}$ . The proof will be given at the end of this section.

In [CMV], the authors established a parabolic Carleman estimate for a class of degenerated operators, in the spirit of [FI], with a weight linked to geodesic distance to the singularity  $\{x = 0\}$ , that is a weight of the form

$$\tilde{\varphi}(x, t) = \frac{x^{2-\alpha} - 1}{(t(T-t))^4} . \quad (2.2.4)$$

In the present article, the design of the weight function  $\varphi$  is similar to (2.2.4). However, as we have to deal with an additional variable  $s$  (see the operator (2.2.1)), we also weaken the weight function in the  $s$  direction (see the weight (2.2.2) which is anisotropic with respect to powers of the Carleman large parameter  $\tau$ ).

## 2.3 Inequality with weight for a specific sum of eigenfunctions

A classical trick on quantitative uniqueness consists on transferring properties for elliptic equation into an estimate for parabolic operator (see [Li]). Here, we naturally reproduce this idea for the sum of eigenfunctions (see [L], [JL], [LR], [LZ], [CSL], [Lu], [LRL], [Le]).

We define the following function space, depending on the frequency parameter  $\Lambda \geq 1$ ,

$$\mathcal{X}_\Lambda := \left\{ \mathbf{u}(s, x) = \sum_{\lambda_j \leq \Lambda} \frac{\sinh(\sqrt{\lambda_j}(s + S_0))}{\sqrt{\lambda_j}} a_j \Phi_j(x); a_j \in \mathbb{R} \right\} .$$

We then go back to a weighted estimate for functions  $\mathbf{u} \in \mathcal{X}_\Lambda$ . Notice that  $\mathcal{Q}\mathbf{u} = 0$ .

Corollary 2.1 .- Let  $\gamma > 0$  defined in (2.2.3). There exist  $\tau_0 > 0$  and  $c > 0$  such that

$$\tau^\gamma \|e^\varphi \mathbf{u}\|_{L^2(Z)}^2 + \tau \int_Z x^\alpha |e^\varphi \partial_x \mathbf{u}|^2 + \tau^3 \int_Z x^{2-\alpha} |e^\varphi \mathbf{u}|^2 \leq c\tau \int_{-S_0}^{S_0} |e^\varphi \partial_x \mathbf{u}|_{x=1}|^2 ,$$

for  $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ , and for all  $\mathbf{u} \in \mathcal{X}_\Lambda$ ,  $\Lambda \geq 1$ .

Proof .- We shall apply the Carleman estimate in Theorem 2.1 to  $v = e^\varphi \mathbf{u}$ , with  $\mathbf{u} \in \mathcal{X}_\Lambda$ . Recall that  $\mathcal{Q}\mathbf{u} = 0$ . Clearly, we have  $v|_{s=-S_0} = \partial_x v|_{s=-S_0} = 0$ , and also  $\mathcal{Q}_\varphi v = 0$ . By Theorem 2.1, this yields

$$\tau^\gamma \|v\|_{L^2(Z)}^2 + \tau \int_Z x^\alpha |\partial_x v|^2 + \tau^3 \int_Z x^{2-\alpha} |v|^2 + \mathcal{B}(v) \leq 0 , \quad (2.3.1)$$

with

$$\begin{aligned} \frac{1}{2}\mathcal{B}(v) \geq & -\tau \int_{-S_0}^{S_0} |\partial_x v|_{x=1}|^2 + 2\frac{\tau^{\gamma/3}}{\nu} \int_0^1 S_0 |\partial_s v|_{s=S_0}|^2 + 2\frac{\tau^{\gamma/3}}{\nu} \int_0^1 v|_{s=S_0} \partial_s v|_{s=S_0} \\ & + 8\frac{\tau^\gamma}{\nu^3} \int_0^1 S_0^3 |v|_{s=S_0}|^2 - 2\tau \int_0^1 x \partial_s v|_{s=S_0} \partial_x v|_{s=S_0} - \tau \int_0^1 v|_{s=S_0} \partial_s v|_{s=S_0} \\ & - 2\frac{\tau^{\gamma/3}}{\nu_0} \int_0^1 x^\alpha S_0 |\partial_x v|_{s=S_0}|^2 + 2\frac{\tau^{2+\gamma/3}}{\nu_0} \int_0^1 x^{2-\alpha} S_0 |v|_{s=S_0}|^2 . \end{aligned} \quad (2.3.2)$$

We first work with volumic terms (from now, the notation  $A \lesssim B$  means that there exists a constant  $c > 0$ , independent on the concerned parameters such that  $A \leq cB$ ). We have

$$\tau \int_Z x^\alpha |e^\varphi \partial_x \mathbf{u}|^2 \lesssim \tau \int_Z x^\alpha |\partial_x v|^2 + \tau \int_Z x^\alpha |(\partial_x \varphi) v|^2 \lesssim \tau \int_Z x^\alpha |\partial_x v|^2 + \tau^3 \int_Z x^{2-\alpha} |v|^2 . \quad (2.3.3)$$

Therefore, from (2.3.1), there exists  $c > 0$  such that

$$\tau^\gamma \|e^\varphi \mathbf{u}\|_{L^2(Z)}^2 + \tau \int_Z x^\alpha |e^\varphi \partial_x \mathbf{u}|^2 + \tau^3 \int_Z x^{2-\alpha} |e^\varphi \mathbf{u}|^2 + c\mathcal{B}(v) \leq 0 . \quad (2.3.4)$$

Remark that

$$\tau^\gamma \|v\|_{L^2(Z)}^2 \geq \tau^\gamma \|v\|_{L^2(Y)}^2 \gtrsim \tau^\gamma e^{-\frac{2}{\nu}\tau^{\gamma/3}s_0^2} \int_0^1 |e^{\tau\frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2 , \quad (2.3.5)$$

and then using (2.3.5) with (2.3.4), we obtain that there exists  $c > 0$  such that

$$\begin{aligned} & e^{-\frac{2}{\nu}\tau^{\gamma/3}s_0^2} \int_0^1 |e^{\tau\frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j(x)|^2 \\ & + \tau^\gamma \|e^\varphi \mathbf{u}\|_{L^2(Z)}^2 + \tau \int_Z x^\alpha |e^\varphi \partial_x \mathbf{u}|^2 + \tau^3 \int_Z x^{2-\alpha} |e^\varphi \mathbf{u}|^2 + c\mathcal{B}(v) \leq 0 . \end{aligned} \quad (2.3.6)$$

Second, we handle the boundary terms. We have, using Young inequality,

$$\tau \left| \int_0^1 x \partial_s v|_{s=S_0} \partial_x v|_{s=S_0} \right| \lesssim \tau^{\gamma/3} \int_0^1 x^\alpha |\partial_x v|_{s=S_0}|^2 + \tau^{2\gamma/3} \int_0^1 x^{2-\alpha} |\partial_s v|_{s=S_0}|^2 .$$

Note that from the form of  $v$ , we obtain that

$$\begin{aligned}
& \tau^{\gamma/3} \int_0^1 x^\alpha |\partial_x v|_{s=S_0}|^2 \\
& \lesssim \tau^\gamma \int_0^1 x^{2-\alpha} |v|_{s=S_0}|^2 + \tau^{\gamma/3} \int_0^1 x^\alpha |e^\varphi \partial_x \mathbf{u}|_{s=S_0}|^2 \\
& \lesssim \tau^\gamma \int_0^1 x^{2-\alpha} |v|_{s=S_0}|^2 + \tau^{\gamma/3} \Lambda \int_0^1 |v|_{s=S_0}|^2 \\
& \lesssim e^{-\frac{2}{\nu} \tau^{\gamma/3} S_0^2} e^{cS_0 \sqrt{\Lambda}} \left( \tau^\gamma \int_0^1 x^{2-\alpha} |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2 + \tau^{\gamma/3} \Lambda \int_0^1 |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2 \right).
\end{aligned}$$

Taking  $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ , with  $\tau_0 > 0$  sufficiently large, yields

$$\tau^{\gamma/3} \int_0^1 x^\alpha |\partial_x v|_{s=S_0}|^2 \lesssim \tau_0^\gamma \Lambda^{3/2} e^{-\frac{2}{\nu} \tau_0^{\gamma/3} \sqrt{\Lambda} S_0^2} e^{cS_0 \sqrt{\Lambda}} \int_0^1 |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2.$$

Also, using the form of  $v$ , and then taking  $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ , with  $\tau_0 > 0$  sufficiently large, one can deduce that

$$\begin{aligned}
\tau^{2\gamma/3} \int_0^1 x^{2-\alpha} |\partial_s v|_{s=S_0}|^2 & \lesssim \tau^\gamma \int_0^1 x^{2-\alpha} |v|_{s=S_0}|^2 + \tau^{2\gamma/3} \int_0^1 x^{2-\alpha} |e^\varphi \partial_s \mathbf{u}|_{s=S_0}|^2 \\
& \lesssim e^{-\frac{2}{\nu} \tau^{\gamma/3} S_0^2} e^{cS_0 \sqrt{\Lambda}} (\tau^\gamma + \tau^{2\gamma/3} \Lambda) \int_0^1 x^{2-\alpha} |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2 \\
& \lesssim \tau_0^\gamma \Lambda^2 e^{-\frac{2}{\nu} \tau_0^{\gamma/3} \sqrt{\Lambda} S_0^2} e^{cS_0 \sqrt{\Lambda}} \int_0^1 |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2.
\end{aligned}$$

Using the same arguments, taking  $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ , with  $\tau_0 > 0$  sufficiently large, yields

$$\tau \left| \int_0^1 v|_{s=S_0} \partial_s v|_{s=S_0} \right| \lesssim \tau_0^\gamma \Lambda^2 e^{-\frac{2}{\nu} \tau_0^{\gamma/3} \sqrt{\Lambda} S_0^2} e^{cS_0 \sqrt{\Lambda}} \int_0^1 |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2.$$

At this point, we see that all the negative terms at  $s = S_0$  in (2.3.2) are bounded by

$$\tau_0^\gamma \Lambda^2 e^{-\frac{2}{\nu} \tau_0^{\gamma/3} \sqrt{\Lambda} S_0^2} e^{cS_0 \sqrt{\Lambda}} \int_0^1 |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2. \quad (2.3.7)$$

As a result, since  $s_0 < S_0$ , the quantity (2.3.7) can be dominated by the first term in the left hand side of (2.3.6), by taking  $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ , with  $\tau_0$  sufficiently large. Hence, from the boundary terms  $\mathcal{B}(v)$ , it only remains  $-\tau \int_{-S_0}^{S_0} |\partial_x v|_{x=1}|^2$ . But,

$$-\tau \int_{-S_0}^{S_0} |\partial_x v|_{x=1}|^2 = -\tau \int_{-S_0}^{S_0} |e^\varphi \partial_x \mathbf{u}|_{x=1}|^2,$$

by using the boundary conditions. This ends the proof.  $\square$



## 2.4 Proof of Theorems 1.1 and 2.1

### 2.4.1 Proof of the spectral inequality

This section is devoted to proving Theorem 1.2. First we establish the spectral inequality with an observation at the boundary  $\{x = 1\}$  by applying Corollary 2.1, we have

$$\tau^\gamma \|e^\varphi \mathbf{u}\|_{L^2(Z)}^2 + \tau \int_Z x^\alpha |e^\varphi \partial_x \mathbf{u}|^2 + \tau^3 \int_Z x^{2-\alpha} |e^\varphi \mathbf{u}|^2 \leq c\tau \int_{-S_0}^{S_0} |e^\varphi \partial_x \mathbf{u}|_{x=1}|^2 ,$$

with for all  $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ , and for all  $u \in X_\Lambda$ ,  $\Lambda \geq 1$ . Arguing as in (2.3.5), we obtain

$$\tau^\gamma \|e^\varphi \mathbf{u}\|_{L^2(Z)}^2 \geq \tau^\gamma \|e^\varphi \mathbf{u}\|_{L^2(Y)}^2 \gtrsim \tau^\gamma e^{-\frac{2}{\nu} \tau^{\gamma/3} s_0^2} \int_0^1 |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \leq \Lambda} a_j \Phi_j|^2 .$$

Bounding the weight functions, and keeping in mind that  $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ , one can deduce that there exist  $C_1, C_2, C_3 > 0$  such that

$$e^{-C_1 \sqrt{\Lambda}} \left\| \sum_{\lambda_j \leq \Lambda} a_j \Phi_j \right\|_{L^2(0,1)}^2 \leq C_2 e^{C_3 \Lambda^{3/(2\gamma)}} \left| \sum_{\lambda_j \leq \Lambda} a_j \Phi_j'(1) \right|^2 ,$$

which is the spectral inequality with a boundary observation. Then, as the region  $\{x = 1\}$  is away from the singularity, the operator  $\mathcal{P}$  is uniformly elliptic there, and therefore it is classical (see for instance [R], [LRL], [L]) that we can propagate the observation  $\{x = 1\}$  to  $\{s = -S_0\} \times \omega$  by using classical Carleman estimates to obtain the desired spectral inequality.  $\square$

### 2.4.2 Proof of Theorem 2.1

Here, we give the proof of the global Carleman estimate near the degeneracy in Theorem 2.1.

Recall that  $\mathcal{Q}_\varphi = e^\varphi \mathcal{Q} e^{-\varphi}$  and therefore

$$\begin{aligned} \mathcal{Q}_\varphi &= -(\partial_s - (\partial_s \varphi))^2 + (\partial_x - (\partial_x \varphi)) x^\alpha (\partial_x - (\partial_x \varphi)) \\ &= -\partial_s^2 - |\partial_s \varphi|^2 + 2(\partial_s \varphi) \partial_s + \partial_s^2 \varphi + \mathcal{P} - x^{2\alpha} |\partial_x \varphi|^2 + 2x^\alpha (\partial_x \varphi) \partial_x + \partial_x (x^\alpha \partial_x \varphi) . \end{aligned}$$

Now, we decompose  $\mathcal{Q}_\varphi$  into four parts:

$$\mathcal{Q}_\varphi = \mathcal{S}_x + \mathcal{S}_s + \mathcal{A}_x + \mathcal{A}_s ,$$

where  $\mathcal{S}_x + \mathcal{S}_s$  is the symmetric part and  $\mathcal{A}_x + \mathcal{A}_s$  is the skew-symmetric part of the full conjugated operator. Using the definition of the weight function (2.2.2), we have

$$\mathcal{S}_x = \mathcal{P} - \tau^2 x^{2-\alpha} , \quad \mathcal{S}_s = -\partial_s^2 - 4 \frac{\tau^{2\gamma/3}}{\nu^2} s^2 , \quad \mathcal{A}_x = 2\tau x \partial_x + \tau , \quad \mathcal{A}_s = -4 \frac{\tau^{\gamma/3}}{\nu} s \partial_s - 2 \frac{\tau^{\gamma/3}}{\nu} .$$

Let  $v \in C^\infty((-S_0, S_0), D(\mathcal{P}))$ . We begin by noting that

$$\begin{aligned} \|\mathcal{Q}_\varphi v\|_{L^2(Z)}^2 &= \|\mathcal{S}v\|_{L^2(Z)}^2 + \|\mathcal{A}v\|_{L^2(Z)}^2 + 2(\mathcal{S}v, \mathcal{A}v)_Z \\ &\geq \|\mathcal{S}v\|_{L^2(Z)}^2 + 2[(\mathcal{S}_x v, \mathcal{A}_x v)_Z + (\mathcal{S}_s v, \mathcal{A}_s v)_Z + (\mathcal{S}_x v, \mathcal{A}_s v)_Z + (\mathcal{S}_s v, \mathcal{A}_x v)_Z] . \end{aligned}$$

The proof is divided into three steps. Each step corresponds to the computation of one of the above scalar products.

*First Step. We begin with the first scalar product  $(\mathcal{S}_x v, \mathcal{A}_x v)_Z$ .*

Lemma 2.3 .- *We have*

$$(\mathcal{S}_x v, \mathcal{A}_x v)_Z = \tau(2 - \alpha) \int_Z x^\alpha |\partial_x v|^2 + \tau^3(2 - \alpha) \int_Z x^{2-\alpha} |v|^2 + \mathcal{B}_0(v) , \quad (4.1)$$

with

$$\mathcal{B}_0(v) = -\tau \int_{-S_0}^{S_0} [x^{\alpha+1} |\partial_x v|^2]_{x=0}^{x=1} - \tau \int_{-S_0}^{S_0} [x^\alpha v \partial_x v]_{x=0}^{x=1} - \tau^3 \int_{-S_0}^{S_0} [x^{3-\alpha} |v|^2]_{x=0}^{x=1} .$$

The proof of this lemma will be provided later. Using the Hardy inequality of Lemma 2.1 in (4.1), there exists  $c > 0$  such that for all  $\alpha \in (0, 2) \setminus \{1\}$ ,

$$c(\mathcal{S}_x v, \mathcal{A}_x v)_Z \geq \tau^2 \int_Z |v|^2 + \tau(2 - \alpha) \int_Z x^\alpha |\partial_x v|^2 + \tau^3(2 - \alpha) \int_Z x^{2-\alpha} |v|^2 + \mathcal{B}_0(v) .$$

In the particular case  $\alpha = 1$ , using the Hardy inequality, for all  $\alpha' \in (1, 2)$ , there exists  $c' > 0$  such that

$$\tau \int_Z x |\partial_x v|^2 \geq \tau \int_Z x^{\alpha'} |\partial_x v|^2 \geq c' \tau \int_Z x^{\alpha'-2} |v|^2 . \quad (4.2)$$

As a result, interpolating (4.2) with (4.1), for all  $\gamma \in (0, 2)$ , there exists  $c' > 0$  such that

$$c'(\mathcal{S}_x v, \mathcal{A}_x v)_Z \geq \tau^\gamma \int_Z |v|^2 + \tau(2 - \alpha) \int_Z x^\alpha |\partial_x v|^2 + \tau^3(2 - \alpha) \int_Z x^{2-\alpha} |v|^2 + \mathcal{B}_0(v) . \quad (4.3)$$

Hence, (4.3) holds for all  $\alpha \in (0, 2)$ , with  $\gamma$  defined in (2.2.3). We now focus on boundary terms  $\mathcal{B}_0$ . We have, using boundary conditions described in  $H_\alpha^1(0, 1)$  and Lemma 2.2,

$$\mathcal{B}_0(v) = -\tau \int_{-S_0}^{S_0} |\partial_x v|_{x=1}|^2 .$$

*Second step. We then compute the second scalar product  $(\mathcal{S}_s v, \mathcal{A}_s v)_Z$ .*

Lemma 2.4 .- *We have*

$$(\mathcal{S}_s v, \mathcal{A}_s v)_Z = -4 \frac{\tau^{\gamma/3}}{\nu} \int_Z |\partial_s v|^2 - 16 \frac{\tau^\gamma}{\nu^3} \int_Z s^2 |v|^2 + \mathcal{B}_1(v) ,$$

with

$$\mathcal{B}_1(v) := 2 \frac{\tau^{\gamma/3}}{\nu} \int_0^1 [s |\partial_s v|^2]_{s=-S_0}^{s=S_0} + 2 \frac{\tau^{\gamma/3}}{\nu} \int_0^1 [v \partial_s v]_{s=-S_0}^{s=S_0} + 8 \frac{\tau^\gamma}{\nu^3} \int_0^1 [s^3 |v|^2]_{s=-S_0}^{s=S_0} .$$

The proof of this lemma will be provided later. The two volumic terms in Lemma 2.4 are non-positive, and need a particular attention. We set

$$K_1(v) := -4 \frac{\tau^{\gamma/3}}{\nu} \int_Z |\partial_s v|^2, \quad K_2(v) := -16 \frac{\tau^\gamma}{\nu^3} \int_Z s^2 |v|^2.$$

Introduce  $\mathcal{S} = \mathcal{S}_x + \mathcal{S}_s$ , we have the following relation

$$- \int_Z |\partial_s v|^2 = \int_Z (\mathcal{S}v)v - 4 \frac{\tau^{2\gamma/3}}{\nu^2} \int_Z s^2 |v|^2 + \int_Z (\mathcal{P}v)v - \tau^2 \int_Z x^{2-\alpha} |v|^2,$$

and we then deduce

$$\begin{aligned} K_1(v) &= -4 \frac{\tau^{\gamma/3}}{\nu} \int_Z (\mathcal{S}v)v - 16 \frac{\tau^\gamma}{\nu^3} \int_Z s^2 |v|^2 + 4 \frac{\tau^{\gamma/3}}{\nu} \int_Z (\mathcal{P}v)v - 4 \frac{\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} |v|^2 \\ &= -4 \frac{\tau^{\gamma/3}}{\nu} \int_Z (\mathcal{S}v)v + K_2(v) + 4 \frac{\tau^{\gamma/3}}{\nu} \int_Z (\mathcal{P}v)v - 4 \frac{\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} |v|^2. \end{aligned}$$

As a result, using integration by parts and Young inequality,

$$\begin{aligned} &(\mathcal{S}_s v, \mathcal{A}_s v)_Z \\ &= -4 \frac{\tau^{\gamma/3}}{\nu} \int_Z (\mathcal{S}v)v + K_2(v) + 4 \frac{\tau^{\gamma/3}}{\nu} \int_Z (\mathcal{P}v)v - 4 \frac{\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} |v|^2 + \mathcal{B}_1(v) \\ &\geq -\frac{2}{\nu} \|\mathcal{S}v\|_{L^2(Z)}^2 - 2 \frac{\tau^{2\gamma/3}}{\nu} \|v\|_{L^2(Z)}^2 + K_2(v) + 4 \frac{\tau^{\gamma/3}}{\nu} \int_Z x^\alpha |\partial_x v|^2 - 4 \frac{\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} |v|^2 \\ &\quad + \tilde{\mathcal{B}}_1(v), \end{aligned}$$

with

$$\tilde{\mathcal{B}}_1(v) = \mathcal{B}_1(v) - \frac{4\tau^{\gamma/3}}{\nu} \int_{-S_0}^{S_0} [x^\alpha v \partial_x v]_{x=0}^{x=1}.$$

Note that, using boundary conditions, we have  $\tilde{\mathcal{B}}_1 = \mathcal{B}_1$ . Summing up, fixing  $\nu := \nu_0 > 0$  sufficiently large, and taking  $\tau \geq \tau_0$ , with  $\tau_0 > 0$  sufficiently large, there exists  $c > 0$  such that

$$\begin{aligned} &c \left( \|\mathcal{S}v\|_{L^2(Z)}^2 + 2(\mathcal{S}_x v, \mathcal{A}_x v) + 2(\mathcal{S}_s v, \mathcal{A}_s v) \right) \\ &\geq \tau^\gamma \|v\|_{L^2(Z)}^2 + \tau \int_Z x^\alpha |\partial_x v|^2 + \tau^3 \int_Z x^{2-\alpha} |v|^2 + 2\mathcal{B}_0(v) + 2\mathcal{B}_1(v). \end{aligned}$$

*Third step. It remains to estimate the crossed-terms  $(\mathcal{S}_x v, \mathcal{A}_s v)_Z + (\mathcal{S}_s v, \mathcal{A}_x v)_Z$ .*

Lemma 2.5 .- *We have, on the one hand*

$$\begin{aligned} (\mathcal{S}_s v, \mathcal{A}_x v)_Z = \mathcal{B}_2(v) := & \tau \int_{-S_0}^{S_0} [x |\partial_s v|^2]_{x=0}^{x=1} - 2\tau \int_0^1 [x \partial_s v \partial_x v]_{s=-S_0}^{s=S_0} \\ & - \tau \int_0^1 [v \partial_s v]_{s=-S_0}^{s=S_0} - 4 \frac{\tau^{1+2\gamma/3}}{\nu_0^2} \int_{-S_0}^{S_0} [s^2 x |v|^2]_{x=0}^{x=1}, \end{aligned}$$

*and on the other hand*

$$\begin{aligned} (\mathcal{S}_x v, \mathcal{A}_s v)_Z = \mathcal{B}_3(v) := & 4 \frac{\tau^{\gamma/3}}{\nu_0} \int_{-S_0}^{S_0} [x^\alpha s \partial_x v \partial_s v]_{x=0}^{x=1} - 2 \frac{\tau^{\gamma/3}}{\nu_0} \int_0^1 [x^\alpha s |\partial_x v|^2]_{s=-S_0}^{s=S_0} \\ & + 2 \frac{\tau^{\gamma/3}}{\nu_0} \int_{-S_0}^{S_0} [x^\alpha v \partial_x v]_{x=0}^{x=1} + 2 \frac{\tau^{2+\gamma/3}}{\nu_0} \int_0^1 [x^{2-\alpha} s |v|^2]_{s=-S_0}^{s=S_0}. \end{aligned}$$

The proof of this lemma will be provided later. Note that using boundary conditions given in  $H_\alpha^1(0, 1)$  as well as Lemma 2.2, we have

$$\mathcal{B}_2(v) \geq -2\tau \int_0^1 [x \partial_s v \partial_x v]_{s=-S_0}^{s=S_0} - \tau \int_0^1 [v \partial_s v]_{s=-S_0}^{s=S_0} ,$$

and

$$\mathcal{B}_3(v) = -2 \frac{\tau^{\gamma/3}}{\nu_0} \int_0^1 [x^\alpha s |\partial_x v|^2]_{s=-S_0}^{s=S_0} + 2 \frac{\tau^{2+\gamma/3}}{\nu_0} \int_0^1 [x^{2-\alpha} s |v|^2]_{s=-S_0}^{s=S_0} .$$

Now setting  $\mathcal{B} = 2(\mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3)$  yields the sought result.  $\square$

### 2.4.3 Proof of Lemma 2.3

We recall that

$$\mathcal{S}_x = \mathcal{P} - \tau^2 x^{2-\alpha} , \quad \mathcal{A}_x = 2\tau x \partial_x + \tau .$$

We shall denote by  $\mathcal{I}_{ij}$  the scalar product between the  $i^{th}$  term of  $\mathcal{S}_x$  with the  $j^{th}$  term of  $\mathcal{A}_x$ . Let us compute first

$$\begin{aligned} \mathcal{I}_{11} &= -2\tau \int_Z \partial_x x^\alpha \partial_x v x \partial_x v \\ &= 2\tau \int_Z x^\alpha |\partial_x v|^2 + \tau \int_Z x^{1+\alpha} \partial_x (|\partial_x v|^2) - 2\tau \int_{-S_0}^{S_0} [x^{1+\alpha} |\partial_x v|^2]_{x=0}^{x=1} \\ &= (1-\alpha)\tau \int_Z x^\alpha |\partial_x v|^2 - \tau \int_{-S_0}^{S_0} [x^{1+\alpha} |\partial_x v|^2]_{x=0}^{x=1} . \end{aligned}$$

Second, we have

$$\mathcal{I}_{12} = -\tau \int_Z v \partial_x x^\alpha \partial_x v = \tau \int_Z x^\alpha |\partial_x v|^2 - \tau \int_{-S_0}^{S_0} [x^\alpha v \partial_x v]_{x=0}^{x=1} .$$

Third, we see that

$$\mathcal{I}_{21} = -2\tau^3 \int_Z x^{3-\alpha} v \partial_x v = -\tau^3 \int_Z x^{3-\alpha} \partial_x (|v|^2) = (3-\alpha)\tau^3 \int_Z x^{2-\alpha} |v|^2 - \tau^3 \int_{-S_0}^{S_0} [x^{3-\alpha} |v|^2]_{x=0}^{x=1} .$$

Finally, we can check that

$$\mathcal{I}_{22} = -\tau^3 \int_Z x^{2-\alpha} |v|^2 ,$$

and we end the proof of Lemma 2.3 by summing the above four quantities.  $\square$

### 2.4.4 Proof of Lemma 2.4

We recall that

$$\mathcal{S}_s = -\partial_s^2 - 4 \frac{\tau^{2\gamma/3}}{\nu^2} s^2 , \quad \mathcal{A}_s = -4 \frac{\tau^{\gamma/3}}{\nu} s \partial_s - 2 \frac{\tau^{\gamma/3}}{\nu} .$$

We shall denote by  $\mathcal{I}_{ij}$  the scalar product between the  $i^{th}$  term of  $\mathcal{S}_s$  with the  $j^{th}$  term of  $\mathcal{A}_s$ . Let us compute the  $\mathcal{I}_{ij}$ ,  $1 \leq i, j \leq 2$ , by integrations by parts

$$\begin{aligned}\mathcal{I}_{11} &= \frac{4\tau^{\gamma/3}}{\nu} \int_Z s \partial_s^2 v s \partial_s v = \frac{2\tau^{\gamma/3}}{\nu} \int_Z s \partial_s (|\partial_s v|^2) = -\frac{2\tau^{\gamma/3}}{\nu} \int_Z |\partial_s v|^2 + \frac{2\tau^{\gamma/3}}{\nu} \int_0^1 [s |\partial_s v|^2]_{s=-S_0}^{s=S_0} , \\ \mathcal{I}_{12} &= \frac{2\tau^{\gamma/3}}{\nu} \int_Z v \partial_s^2 v = -\frac{2\tau^{\gamma/3}}{\nu} \int_Z |\partial_s v|^2 + \frac{2\tau^{\gamma/3}}{\nu} \int_0^1 [v \partial_s v]_{s=-S_0}^{s=S_0} , \\ \mathcal{I}_{21} &= \frac{16\tau^\gamma}{\nu^3} \int_Z s^3 v \partial_s v = 8 \frac{\tau^\gamma}{\nu^3} \int_Z s^3 \partial_s (|v|^2) = -\frac{24\tau^\gamma}{\nu^3} \int_Z s^2 |v|^2 + \frac{8\tau^\gamma}{\nu^3} \int_0^1 [s^3 |v|^2]_{s=-S_0}^{s=S_0} ,\end{aligned}$$

and

$$\mathcal{I}_{22} = 8 \frac{\tau^\gamma}{\nu^3} \int_Z s^2 |v|^2 .$$

Summing all the  $\mathcal{I}_{ij}$  yields the sought result of Lemma 2.4.  $\square$

### 2.4.5 Proof of Lemma 2.5

We recall that

$$\mathcal{S}_x = \mathcal{P} - \tau^2 x^{2-\alpha} , \quad \mathcal{S}_s = -\partial_s^2 - 4 \frac{\tau^{2\gamma/3}}{\nu^2} s^2 , \quad \mathcal{A}_x = 2\tau x \partial_x + \tau , \quad \mathcal{A}_s = -4 \frac{\tau^{\gamma/3}}{\nu} s \partial_s - 2 \frac{\tau^{\gamma/3}}{\nu} .$$

We first compute the scalar product  $(\mathcal{S}_s v, \mathcal{A}_x v)_Z$ . We shall denote by  $\mathcal{I}_{ij}$  the scalar product between the  $i^{th}$  term of  $\mathcal{S}_s$  with the  $j^{th}$  term of  $\mathcal{A}_x$ . We have

$$\begin{aligned}\mathcal{I}_{11} &= -2\tau \int_Z x \partial_s^2 v \partial_x v = \tau \int_Z x \partial_x (|\partial_s v|^2) - 2\tau \int_0^1 [x \partial_s v \partial_x v]_{s=-S_0}^{s=S_0} \\ &= -\tau \int_Z |\partial_s v|^2 + \tau \int_{-S_0}^{S_0} [x |\partial_s v|^2]_{x=0}^{x=1} - 2\tau \int_0^1 [x \partial_s v \partial_x v]_{s=-S_0}^{s=S_0} , \\ \mathcal{I}_{12} &= -\tau \int_Z v \partial_s^2 v = \tau \int_Z |\partial_s v|^2 - \tau \int_0^1 [v \partial_s v]_{s=-S_0}^{s=S_0} , \\ \mathcal{I}_{21} &= -\frac{8\tau^{2\gamma/3+1}}{\nu^2} \int_Z s^2 x v \partial_x v = -\frac{4\tau^{2\gamma/3+1}}{\nu^2} \int_Z s^2 x \partial_x (|v|^2) \\ &= \frac{4\tau^{2\gamma/3+1}}{\nu^2} \int_Z s^2 |v|^2 - \frac{4\tau^{2\gamma/3+1}}{\nu^2} \int_{-S_0}^{S_0} [s^2 x |v|^2]_{x=0}^{x=1} ,\end{aligned}$$

and

$$\mathcal{I}_{22} = \frac{4\tau^{2\gamma/3+1}}{\nu^2} \int_Z s^2 |v|^2 .$$

Summing the above quantities yields the result, by remarking that all the volumic terms cancel. We second compute the scalar product  $(\mathcal{S}_x v, \mathcal{A}_s v)_Z$ . We shall denote by  $\mathcal{J}_{ij}$  the scalar product between the  $i^{th}$  term of  $\mathcal{S}_x$  with the  $j^{th}$  term of  $\mathcal{A}_s$ . Integrations by parts then give

$$\begin{aligned}\mathcal{J}_{11} &= \frac{4\tau^{\gamma/3}}{\nu} \int_Z s \partial_x (x^\alpha \partial_x v) \partial_s v = -\frac{2\tau^{\gamma/3}}{\nu} \int_Z x^\alpha s \partial_s (|\partial_x v|^2) + \frac{4\tau^{\gamma/3}}{\nu} \int_{-S_0}^{S_0} [x^\alpha s \partial_x v \partial_s v]_{x=0}^{x=1} \\ &= \frac{2\tau^{\gamma/3}}{\nu} \int_Z x^\alpha |\partial_x v|^2 - \frac{2\tau^{\gamma/3}}{\nu} \int_0^1 [x^\alpha s |\partial_x v|^2]_{s=-S_0}^{s=S_0} + \frac{4\tau^{\gamma/3}}{\nu} \int_{-S_0}^{S_0} [x^\alpha s \partial_x v \partial_s v]_{x=0}^{x=1} ,\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{12} &= \frac{2\tau^{\gamma/3}}{\nu} \int_Z v \partial_x (x^\alpha \partial_x v) = -\frac{2\tau^{\gamma/3}}{\nu} \int_Z x^\alpha |\partial_x v|^2 + \frac{2\tau^{\gamma/3}}{\nu} \int_{-S_0}^{S_0} [x^\alpha v \partial_x v]_{x=0}^{x=1} , \\
\mathcal{J}_{21} &= \frac{4\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} v s \partial_s v = \frac{2\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} s \partial_s (|v|^2) \\
&= -\frac{2\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} |v|^2 + \frac{2\tau^{2+\gamma/3}}{\nu} \int_0^Z [x^{2-\alpha} s |v|^2]_{s=-S_0}^{s=S_0} ,
\end{aligned}$$

and

$$\mathcal{J}_{22} = \frac{2\tau^{2+\gamma/3}}{\nu} \int_Z x^{2-\alpha} |v|^2 .$$

It remains to sum the above  $\mathcal{J}_{ij}$  to obtain the sought result of Lemma 2.5.  $\square$

### 3 Applications of spectral inequality

The second part of this article is devoted to show some applications of the spectral inequality.

Let  $H$  be a real Hilbert space, and  $P$  a linear self-adjoint operator from  $D(P)$  into  $H$ , where  $D(P)$  being the domain of  $P$  is a subspace of  $H$ . Denote by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the norm and the inner product of  $H$  respectively. We assume that  $P$  is an isomorphism from  $D(P)$  (equipped with the graph norm) onto  $H$ , that  $P^{-1}$  is a linear compact operator in  $H$  and that  $\langle P\vartheta, \vartheta \rangle > 0 \ \forall \vartheta \in D(P), \vartheta \neq 0$ . Introduce the set  $\{\lambda_j\}_{j \geq 1}$  for the family of all eigenvalues of  $P$  so that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \lambda_{k+1} \leq \dots \text{ and } \lim_{j \rightarrow \infty} \lambda_j = \infty ,$$

and let  $\{\Phi_j\}_{j \geq 1}$  be the family of the corresponding orthogonal normalized eigenfunctions.

It is well known that for  $u_0 \in H$  given, the initial value problem

$$\begin{cases} u'(t) + Pu(t) = 0 , & t \in (0, +\infty) , \\ u(0) = u_0 , \end{cases}$$

possesses a unique solution  $u \in L^2(0, T; D(P^{1/2})) \cap C([0, T], H)$  for any  $T > 0$  which satisfies

$$u(t) = \sum_{j \geq 1} \langle u_0, \Phi_j \rangle e^{-\lambda_j t} \Phi_j \text{ and } \|u(t)\| \leq e^{-\lambda_1 t} \|u_0\| .$$

In particular, if  $u_0 = \sum_{j \geq 1} a_j \Phi_j$  with  $\sum_{j \geq 1} |a_j|^2 < +\infty$ , then  $\|u_0\|^2 = \sum_{j \geq 1} |a_j|^2$ ,  $\langle Pu_0, u_0 \rangle = \sum_{j \geq 1} \lambda_j |a_j|^2$  and  $\langle P^{-1}u_0, u_0 \rangle = \sum_{j \geq 1} \frac{1}{\lambda_j} |a_j|^2$ . Further,  $\frac{d}{dt} \|u(t)\|^2 + 2 \langle Pu(t), u(t) \rangle = 0$  and  $\frac{d}{dt} \langle P^{-1}u(t), u(t) \rangle + 2 \|u(t)\|^2 = 0$ .

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^d$ ,  $d \geq 1$ , with boundary  $\partial\Omega$  of class  $C^2$ . Four examples of operator  $P$  are the following:

- The 1d degenerated operator with  $d = 1$  and  $P = -\partial_x (x^\alpha \partial_x)$  with  $\Omega = (0, 1)$ ,  $H = L^2(\Omega)$  and  $D(P) = \{\vartheta \in H_\alpha^1(\Omega) ; \mathcal{P}\vartheta \in L^2(\Omega) \text{ and } \text{BC}_\alpha(\vartheta) = 0\}$  ;

- The Laplacian with  $P = -\Delta$  with  $H = L^2(\Omega)$  and  $D(P) = H^2(\Omega) \cap H_0^1(\Omega)$  ;
- The bi-Laplacian with  $P = \Delta^2$  with  $H = L^2(\Omega)$  and  $D(P) = H^4(\Omega) \cap H_0^2(\Omega)$  ;
- The Stokes operator with  $P = -\mathbb{P}\Delta$  with  $H = \left\{ \vartheta \in L^2(\Omega)^d ; \operatorname{div} \vartheta = 0, \vartheta \cdot n|_{\partial\Omega} = 0 \right\}$  and  $D(P) = H^2(\Omega)^d \cap \left\{ \vartheta \in H_0^1(\Omega)^d ; \operatorname{div} \vartheta = 0 \right\}$  where  $\mathbb{P}$  is the orthogonal projector in  $L^2(\Omega)^d$  onto  $H$ .

### 3.1 Equivalence between observation and spectral inequality

In this section, we present several equivalent inequalities. From now, suppose that  $H = L^2(\Omega)$ . Denote  $\|\cdot\|_\omega$  and  $\langle \cdot, \cdot \rangle_\omega$  the norm and the inner product of  $L^2(\omega)$  respectively where  $\omega$  is a subdomain of  $\Omega$ .

**Theorem 3.1 .-** *Let  $\omega$  be an open and nonempty subset of  $\Omega$ . Let  $\sigma \in (0, 1)$ . Then the following statements are equivalent:*

- (i) *There is a positive constant  $C_1$ , depending only on  $P, \Omega, \omega$  and  $\sigma$ , so that for each  $\Lambda > 0$  and each sequence of real numbers  $\{a_j\} \subset \mathbb{R}$ , it holds*

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq e^{C_1(1+\Lambda^\sigma)} \int_\omega \left| \sum_{\lambda_j \leq \Lambda} a_j \Phi_j \right|^2 .$$

- (ii) *There is a positive constant  $C_2$ , depending only on  $(P, \Omega, \omega, \sigma)$ , so that for all  $\theta \in (0, 1)$ ,  $t > 0$  and  $u(0) \in L^2(\Omega)$ ,*

$$\|u(t)\| \leq e^{C_2 \left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} \|u(0)\|^\theta \|u(t)\|_\omega^{1-\theta} .$$

- (iii) *There is a positive constant  $C_3$ , depending only on  $(P, \Omega, \omega, \sigma)$ , so that for all  $\varepsilon > 0$ ,  $t > 0$  and  $u(0) \in L^2(\Omega)$ ,*

$$\|u(t)\|^2 \leq p_\sigma(t, \varepsilon) \|u(t)\|_\omega^2 + \varepsilon \|u(0)\|^2 ,$$

where

$$p_\sigma(t, \varepsilon) = e^{C_3 \left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{t} \ln(e + \frac{1}{\varepsilon})\right)^\sigma} .$$

- (iv) *There is a positive constant  $C_4$ , depending only on  $(P, \Omega, \omega, \sigma)$ , so that for all  $t > 0$  and  $u(0) \in L^2(\Omega)$ ,*

$$\|u(t)\| \leq e^{C_4 \left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_4}{t} \ln\left(\frac{\|u(0)\|}{\|u(t)\|}\right)\right)^\sigma} \|u(t)\|_\omega .$$

In particular, if  $P = -\Delta$ , then  $\sigma = \frac{1}{2}$  (see [L], [LZ], [PWX], [BP], [Ph2]); If  $P = \Delta^2$ , then  $\sigma = \frac{1}{4}$  (see [AE], [EMZ], [Ga], [LRR2]); If  $P$  is the Stokes operator, then  $\sigma = \frac{1}{2}$  (see [CSL]).

Proof .- We organize the proof by several steps.

*Step 1: To show that (i)  $\Rightarrow$  (ii).*

Arbitrarily fix  $\lambda > 0$ ,  $t > 0$  and  $u(0) = \sum_{j \geq 1} a_j e_j$  with  $\{a_j\}_{j \geq 1} \subset \ell^2$ . Write

$$u(t) = \sum_{\lambda_j \leq \Lambda} a_j e^{-\lambda_j t} \Phi_j + \sum_{\lambda_j > \Lambda} a_j e^{-\lambda_j t} \Phi_j .$$

Then by (i), we find that

$$\begin{aligned} \|u(t)\| &\leq \left\| \sum_{\lambda_j \leq \Lambda} a_j e^{-\lambda_j t} \Phi_j \right\| + \left\| \sum_{\lambda_j > \Lambda} a_j e^{-\lambda_j t} \Phi_j \right\| \\ &\leq \left( \sum_{\lambda_j \leq \Lambda} |a_j e^{-\lambda_j t}|^2 \right)^{1/2} + e^{-\Lambda t} \|u(0)\| \\ &\leq \left( e^{C_1(1+\Lambda^\sigma)} \int_{\omega} \left| \sum_{\lambda_j \leq \Lambda} a_j e^{-\lambda_j t} \Phi_j \right|^2 \right)^{1/2} + e^{-\Lambda t} \|u(0)\| . \end{aligned}$$

This, along with the triangle inequality for the norm  $\|\cdot\|_{\omega}$ , yields that

$$\begin{aligned} \|u(t)\| &\leq \left( e^{C_1(1+\Lambda^\sigma)} \int_{\omega} \left| \sum_{j \geq 1} a_j e^{-\lambda_j t} \Phi_j \right|^2 \right)^{1/2} \\ &\quad + \left( e^{C_1(1+\Lambda^\sigma)} \int_{\omega} \left| \sum_{\lambda_j > \Lambda} a_j e^{-\lambda_j t} \Phi_j \right|^2 \right)^{1/2} + e^{-\Lambda t} \|u(0)\| . \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \|u(t)\| &\leq e^{\frac{C_1}{2}(1+\Lambda^\sigma)} \|u(t)\|_{\omega} + e^{\frac{C_1}{2}(1+\Lambda^\sigma)} e^{-\Lambda t} \|u(0)\| + e^{-\Lambda t} \|u(0)\| \\ &\leq 2e^{\frac{C_1}{2}(1+\Lambda^\sigma)} (\|u(t)\|_{\omega} + e^{-\Lambda t} \|u(0)\|) . \end{aligned}$$

Since by the Young inequality

$$C_1 \Lambda^\sigma = \frac{C_1}{(\epsilon t)^\sigma} (\epsilon \Lambda t)^\sigma \leq \epsilon \Lambda t + \left( \frac{C_1}{(\epsilon t)^\sigma} \right)^{\frac{1}{1-\sigma}} \quad \text{for any } \epsilon, t > 0 ,$$

one deduce that for all  $\epsilon \in (0, 2)$ ,

$$\|u(t)\| \leq 2e^{\frac{C_1}{2}} e^{\frac{1}{2}(\frac{C_1}{(\epsilon t)^\sigma})^{\frac{1}{1-\sigma}}} \left( e^{\frac{\epsilon}{2}\Lambda t} \|u(t)\|_{\omega} + e^{-\frac{2-\epsilon}{2}\Lambda t} \|u(0)\| \right) \quad \text{for each } \Lambda > 0 .$$

Notice that if  $\|u(t)\|_{\omega} = 0$  then,  $\|u(t)\| = 0$ . Next, choose

$$\Lambda = \frac{1}{t} \ln \left( \frac{\|u(0)\|}{\|u(t)\|_{\omega}} \right)$$



(knowing that  $\|u(t)\|_\omega \leq \|u(0)\|$ ) to get

$$\|u(t)\| \leq 2e^{\frac{C_1}{2}} e^{\frac{1}{2}(\frac{C_1}{(\varepsilon t)^\sigma})^{\frac{1}{1-\sigma}}} \left( 2\|u(t)\|_\omega^{1-\frac{\varepsilon}{2}} \|u(0)\|^{\frac{\varepsilon}{2}} \right)$$

which is the inequality in (ii) with  $\theta = \frac{\varepsilon}{2}$  and  $\ln 4 + \frac{C_1}{2} + \frac{1}{2} \left( \frac{C_1}{(\varepsilon t)^\sigma} \right)^{\frac{1}{1-\sigma}} \leq C_2 \left( 1 + \left( \frac{1}{\theta t} \right)^{\frac{\sigma}{1-\sigma}} \right)$ .

*Step 2: To show that (ii)  $\Rightarrow$  (iii).*

We write the inequality in (ii) in the following way

$$\|u(t)\|^2 \leq \|u(0)\|^{2\theta} \left( \exp \left( \frac{2C_2}{1-\theta} \left( 1 + \left( \frac{1}{\theta t} \right)^{\frac{\sigma}{1-\sigma}} \right) \right) \|u(t)\|_\omega^2 \right)^{1-\theta}$$

and apply the fact that for any  $E, B, D > 0$  and  $\theta \in (0, 1)$

$$E \leq B^\theta D^{1-\theta} \Leftrightarrow E \leq \varepsilon B + (1-\theta) \theta^{\frac{\theta}{1-\theta}} \frac{1}{\varepsilon^{\frac{\theta}{1-\theta}}} D \quad \forall \varepsilon > 0.$$

To prove the above equivalence, one uses the Young inequality and one choose  $\varepsilon = \theta \left( \frac{D}{B} \right)^{1-\theta}$ . Therefore,

$$\|u(t)\|^2 \leq \varepsilon \|u(0)\|^2 + \exp \left( \frac{2C_2}{1-\theta} \left( 1 + \left( \frac{1}{\theta t} \right)^{\frac{\sigma}{1-\sigma}} \right) \right) \frac{1}{\varepsilon^{\frac{\theta}{1-\theta}}} \|u(t)\|_\omega^2.$$

By denoting  $\beta = \frac{\theta}{1-\theta}$ , it yields

$$\|u(t)\|^2 \leq \varepsilon \|u(0)\|^2 + e^{2C_2(1+\beta) \left( 1 + \left( \frac{1+\beta}{\beta t} \right)^{\frac{\sigma}{1-\sigma}} \right)} \frac{1}{\varepsilon^\beta} \|u(t)\|_\omega^2.$$

Now, notice, with  $B = K \left( 1 + \left( \frac{1}{t} \right)^{\frac{\sigma}{1-\sigma}} \right)$  and  $D = K \left( \frac{1}{t} \right)^{\frac{\sigma}{1-\sigma}}$  for some constant  $K > 0$ , that

$$\frac{1}{\varepsilon^\beta} e^{2C_2(1+\beta) \left( 1 + \left( \frac{1+\beta}{\beta t} \right)^{\frac{\sigma}{1-\sigma}} \right)} \leq e^{B+\beta(\ln(e+\frac{1}{\varepsilon})+B)+(\frac{1}{\beta})^{\frac{\sigma}{1-\sigma}} D}.$$

Next, choose  $\beta = \left( \frac{D}{\ln(e+\frac{1}{\varepsilon})+B} \right)^{1-\sigma}$  to get

$$e^{B+\beta(\ln(e+\frac{1}{\varepsilon})+B)+(\frac{1}{\beta})^{\frac{\sigma}{1-\sigma}} D} \leq e^{cB+c(\ln(e+\frac{1}{\varepsilon})+B)^\sigma D^{1-\sigma}} \leq e^{c'B+c'(\ln(e+\frac{1}{\varepsilon}))^\sigma D^{1-\sigma}}$$

for some constants  $c, c' > 0$ . Therefore, we obtain the desired inequality

$$\|u(t)\|^2 \leq \varepsilon \|u(0)\|^2 + e^{C_3 \left( 1 + \left( \frac{1}{t} \right)^{\frac{\sigma}{1-\sigma}} \right)} e^{\left( \frac{C_3}{t} \ln(e+\frac{1}{\varepsilon}) \right)^\sigma} \|u(t)\|_\omega^2.$$

*Step 3: to show that (iii)  $\Rightarrow$  (iv).*

Take

$$\varepsilon = \frac{1}{2} \frac{\|u(t)\|^2}{\|u(0)\|^2}$$

in the inequality in (iii) and we use the fact that  $\|u(t)\| \leq \|u(0)\|$ . Therefore, we have

$$\frac{1}{2} \|u(t)\|^2 \leq e^{C_3 \left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{t} \ln \left((e+2) \frac{\|u(0)\|^2}{\|u(t)\|^2}\right)\right)^\sigma} \|u(t)\|_\omega^2 .$$

*Step 4: to show that (iv)  $\Rightarrow$  (i).*

Apply the Young inequality

$$\left(\frac{C_4}{t} \ln \left(\frac{\|u(0)\|}{\|u(t)\|}\right)\right)^\sigma \leq \left(\frac{C_4}{t}\right)^{\frac{\sigma}{1-\sigma}} + \ln \left(\frac{\|u(0)\|}{\|u(t)\|}\right)$$

to deduce the inequality: There are two constants  $C > 0$  and  $\alpha \in (0, 1)$ , which depend only on  $(\Omega, \omega, \sigma)$ , so that for all  $t > 0$  and  $u(0) \in L^2(\Omega)$ ,

$$\|u(t)\| \leq e^{C \left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} \|u(0)\|^\alpha \|u(t)\|_\omega^{1-\alpha} .$$

Arbitrarily fix  $\Lambda > 0$  and  $\{a_j\} \subset \mathbb{R}$ . By applying the above inequality, with  $u(0) = \sum_{\lambda_j < \Lambda} a_j e^{\lambda_j t} \Phi_j$ , we get that

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq e^{2C \left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} \left(\sum_{\lambda_j \leq \Lambda} |a_j e^{\lambda_j t}|^2\right)^\alpha \left(\int_\omega \left|\sum_{\lambda_j \leq \Lambda} a_j \Phi_j\right|^2\right)^{1-\alpha} ,$$

which implies that

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq e^{\frac{2}{1-\alpha} C \left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\frac{2\alpha}{1-\alpha} \Lambda t} \int_\omega \left|\sum_{\lambda_j \leq \Lambda} a_j \Phi_j\right|^2 \quad \text{for each } t > 0 .$$

Choose  $t = \left(\frac{1}{\Lambda}\right)^{1-\sigma}$  to get the conclusion (i).

This ends the proof. □

### 3.2 Equivalence between observation and control

Let us recall the classical results of equivalence between observation estimate and controllability with cost. There are at least three ways to establish the cost: One is based on the duality of the control operator in the spirit of the HUM method (see [Lio]) with a spectral decomposition (see [R], [Ph]); Another one have a geometric point of view using Hahn-Banach Theorem (see [WWZ], [WYZ]) ; The last one is based on a minimization of a certain functional (see [FZ], [Mi]). The arguments we present are similar to those appear in [Mi, lemma 3.2, p.1475] (see also [DM, remark 6.6, p.3670]).

Denote  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the norm and the inner product of  $L^2(\Omega)$  respectively.

**Theorem 3.2.-** *Let  $0 \leq T_0 < T_1 < T_2$ . Let  $\ell, \varepsilon > 0$ . The following two statements are equivalent.*

(C) For any  $y_e \in L^2(\Omega)$ , there is  $f \in L^2(\omega)$  such that the solution  $y$  to

$$\begin{cases} y'(t) + Py(t) = 0, & t \in (T_0, T_2) \setminus \{T_1\}, \\ y(T_0) = y_e, \\ y(T_1) = y(T_1-) + 1_\omega f, \end{cases}$$

satisfies

$$\frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y(T_2)\|^2 \leq \|y_e\|^2.$$

(O) The solution  $u$  to

$$\begin{cases} u'(t) + Pu(t) = 0, & t \in (T_0, T_2), \\ u(T_0) \in L^2(\Omega), \end{cases}$$

satisfies

$$\|u(T_2)\|^2 \leq \ell \|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|u(T_0)\|^2.$$

Proof of (C)  $\Rightarrow$  (O) .- We multiply the equations of (C) by  $u(T_0 + T_2 - t)$  to get

$$\langle y(T_2), u(T_0) \rangle - \langle y(T_0), u(T_2) \rangle = \langle f, u(T_0 + T_2 - T_1) \rangle_\omega,$$

that is,

$$\langle y_e, u(T_2) \rangle = -\langle f, u(T_0 + T_2 - T_1) \rangle_\omega + \langle y(T_2), u(T_0) \rangle.$$

By Cauchy-Schwarz inequality and using the inequality in (C) one can deduce that

$$\begin{aligned} \langle y_e, u(T_2) \rangle &\leq \|f\|_\omega \|u(T_0 + T_2 - T_1)\|_\omega + \|y(T_2)\| \|u(T_0)\| \\ &\leq \frac{1}{2\ell} \|f\|_\omega^2 + \frac{1}{2\varepsilon} \|y(T_2)\|^2 + \frac{\ell}{2} \|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2} \|u(T_0)\|^2 \\ &\leq \frac{1}{2} \|y_e\|^2 + \frac{1}{2} (\ell \|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|u(T_0)\|^2) \end{aligned}$$

which gives the desired estimate by choosing  $y_e = u(T_2)$ .

Proof of (O)  $\Rightarrow$  (C) .- Let  $y_e \in L^2(\Omega)$ . Consider the functional  $J$  defined on  $L^2(\Omega)$  given by

$$J(\vartheta) = \frac{\ell}{2} \|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2} \|\vartheta\|^2 - \langle y_e, u(T_2) \rangle,$$

where

$$\begin{cases} u'(t) + Pu(t) = 0, & t \in (T_0, T_2), \\ u(T_0) = \vartheta. \end{cases}$$

Notice that  $J$  is strictly convex,  $C^1$  and coercive and therefore  $J$  has a unique minimizer  $w_0 \in L^2(\Omega)$ , i.e.  $J(w_0) = \min_{\vartheta \in L^2(\Omega)} J(\vartheta)$ . Set

$$\begin{cases} w'(t) + Pw(t) = 0, & t \in (T_0, T_2), \\ w(T_0) = w_0, \end{cases} \quad \text{and} \quad \begin{cases} h'(t) + Ph(t) = 0, & t \in (T_0, T_2), \\ h(T_0) = h_0. \end{cases}$$

Since  $J'(w_0)h_0 = 0$  for any  $h_0 \in L^2(\Omega)$ , we have

$$\ell \langle w(T_0 + T_2 - T_1), h(T_0 + T_2 - T_1) \rangle_\omega + \varepsilon \langle w_0, h_0 \rangle - \langle y_e, h(T_2) \rangle = 0 \quad \forall h_0 \in L^2(\Omega).$$

On the other hand, the identity

$$\langle y(T_2), u(T_0) \rangle - \langle y_e, u(T_2) \rangle = \langle f, u(T_0 + T_2 - T_1) \rangle_\omega \quad \forall u(T_0) \in L^2(\Omega)$$

implies

$$-\langle f, h(T_0 + T_2 - T_1) \rangle_\omega + \langle y(T_2), h_0 \rangle - \langle y_e, h(T_2) \rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

By choosing  $f = -\ell w(T_0 + T_2 - T_1)$ , we deduce that the solution  $y$  satisfies

$$\varepsilon w_0 = y(T_2) .$$

Further,

$$\ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 = \frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y(T_2)\|^2 .$$

Moreover, taking  $h_0 = w_0$  into  $J'(w_0)h_0 = 0$ , we get

$$\ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 - \langle y_e, w(T_2) \rangle = 0 .$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} \ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 &\leq \|y_e\|_{L^2(\Omega)} \|w(T_2)\|_{L^2(\Omega)} \\ &\leq \|y_e\|_{L^2(\Omega)} \left( \ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 \right)^{1/2} \end{aligned}$$

where in the last line, we used  $(\mathcal{O})$ . Therefore, we get

$$\ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 \leq \|y_e\|^2 ,$$

that is,

$$\frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y(T_2)\|^2 \leq \|y_e\|^2$$

where

$$\begin{cases} y'(t) + Py(t) = 0 , & t \in (T_0, T_2) \setminus \{T_1\} , \\ y(T_0) = y_e , \\ y(T_1) = y(T_{1-}) + 1_\omega(-\ell w(T_0 + T_2 - t)) , \\ w'(t) + Pw(t) = 0 , & t \in (T_0, T_2) , \\ w(T_0) = \frac{1}{\varepsilon} y(T_2) . \end{cases}$$

This completes the proof. □

**Theorem 3.3.-** *Let  $0 \leq T_0 < T_1 < T_2$ . Let  $\ell, \varepsilon > 0$ . The following two statements are equivalent.*

**(C)** *For any  $y_d \in L^2(\Omega)$  such that  $\langle Py_d, y_d \rangle < +\infty$ , there is  $f \in L^2(\omega)$  such that the solution  $y$  to*

$$\begin{cases} y'(t) + Py(t) = 0 , & t \in (T_0, T_2) \setminus \{T_1\} , \\ y(T_0) = 0 , \\ y(T_1) = y(T_{1-}) + 1_\omega f , \end{cases}$$

*satisfies*

$$\frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y(T_2) - y_d\|^2 \leq \langle Ay_d, y_d \rangle .$$

( $\mathcal{O}$ ) The solution  $u$  to

$$\begin{cases} u'(t) + Pu(t) = 0, & t \in (T_0, T_2), \\ u(T_0) \in L^2(\Omega), \end{cases}$$

satisfies

$$\langle P^{-1}u(T_0), u(T_0) \rangle \leq \ell \|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|u(T_0)\|^2.$$

Proof of ( $\mathcal{C}$ )  $\Rightarrow$  ( $\mathcal{O}$ ) .- We multiply the equations of ( $\mathcal{C}$ ) by  $u(T_0 + T_2 - t)$  to get

$$\langle y(T_2), u(T_0) \rangle - \langle y(T_0), u(T_2) \rangle = \langle f, u(T_0 + T_2 - T_1) \rangle_\omega,$$

that is,

$$\langle y_d, u(T_0) \rangle = \langle f, u(T_0 + T_2 - T_1) \rangle_\omega - \langle y(T_2) - y_d, u(T_0) \rangle.$$

By Cauchy-Schwarz inequality and using the inequality in ( $\mathcal{C}$ ) one has

$$\begin{aligned} \langle y_d, u(T_0) \rangle &\leq \|f\|_\omega \|u(T_0 + T_2 - T_1)\|_\omega + \|y(T_2) - y_d\| \|u(T_0)\| \\ &\leq \frac{1}{2\ell} \|f\|_\omega^2 + \frac{1}{2\varepsilon} \|y(T_2) - y_d\|^2 + \frac{\ell}{2} \|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2} \|u(T_0)\|^2 \\ &\leq \frac{1}{2} \langle Py_d, y_d \rangle + \frac{1}{2} (\ell \|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|u(T_0)\|^2) \end{aligned}$$

which gives the desired estimate by choosing  $y_d = P^{-1}u(T_0)$ .

Proof of ( $\mathcal{O}$ )  $\Rightarrow$  ( $\mathcal{C}$ ) .- Let  $y_d \in L^2(\Omega)$  such that  $\langle Py_d, y_d \rangle < +\infty$ . Consider the functional  $J$  defined on  $L^2(\Omega)$  given by

$$J(\vartheta) = \frac{\ell}{2} \|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2} \|\vartheta\|^2 + \langle y_d, \vartheta \rangle,$$

where

$$\begin{cases} u'(t) + Pu(t) = 0, & t \in (T_0, T_2), \\ u(T_0) = \vartheta. \end{cases}$$

Notice that  $J$  is strictly convex,  $C^1$  and coercive and therefore  $J$  has a unique minimizer  $w_0 \in L^2(\Omega)$ , i.e.  $J(w_0) = \min_{\vartheta \in L^2(\Omega)} J(\vartheta)$ . Set

$$\begin{cases} w'(t) + Pw(t) = 0, & t \in (T_0, T_2), \\ w(T_0) = w_0, \end{cases} \quad \text{and} \quad \begin{cases} h'(t) + Ph(t) = 0, & t \in (T_0, T_2), \\ h(T_0) = h_0. \end{cases}$$

Since  $J'(w_0)h_0 = 0$  for any  $h_0 \in L^2(\Omega)$ , we have

$$\ell \langle w(T_0 + T_2 - T_1), h(T_0 + T_2 - T_1) \rangle_\omega + \varepsilon \langle w_0, h_0 \rangle + \langle y_d, h_0 \rangle = 0 \quad \forall h_0 \in L^2(\Omega).$$

On the other hand, the identity

$$\langle y(T_2), u(T_0) \rangle - \langle y(T_0), u(T_2) \rangle = \langle f, u(T_0 + T_2 - T_1) \rangle_\omega \quad \forall u(T_0) \in L^2(\Omega)$$

implies

$$-\langle f, h(T_0 + T_2 - T_1) \rangle_\omega + \langle y(T_2) - y_d, h_0 \rangle + \langle y_d, h_0 \rangle = 0 \quad \forall h_0 \in L^2(\Omega).$$

By choosing  $f = -\ell w (T_0 + T_2 - T_1)$ , we deduce that the solution  $y$  satisfies

$$\varepsilon w_0 = y (T_2) - y_d .$$

Further,

$$\ell \|w (T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 = \frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y (T_2) - y_d\|^2 .$$

Moreover, taking  $h_0 = w_0$  into  $J'(w_0)h_0 = 0$ , we get

$$\ell \|w (T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 + \langle y_d, w_0 \rangle = 0 .$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} \ell \|w (T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 &\leq \langle Py_d, y_d \rangle \langle P^{-1}w_0, w_0 \rangle \\ &\leq \langle Py_d, y_d \rangle \left( \ell \|w (T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 \right)^{1/2} \end{aligned}$$

where in the last line, we used  $(\mathcal{O})$ . Therefore, we get

$$\ell \|w (T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 \leq \langle Py_d, y_d \rangle ,$$

that is,

$$\frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y (T_2) - y_d\|^2 \leq \langle Py_d, y_d \rangle$$

where

$$\begin{cases} y' (t) + Py (t) = 0 , & t \in (T_0, T_2) \setminus \{T_1\} , \\ y (T_0) = 0 , \\ y (T_1) = y (T_{1-}) + 1_\omega (-\ell w (T_0 + T_2 - t)) , \\ w' (t) + Pw (t) = 0 , & t \in (T_0, T_2) , \\ w (T_0) = \frac{1}{\varepsilon} (y (T_2) - y_d) . \end{cases}$$

This completes the proof. □

### 3.3 Approximate impulse control

Direct applications of Theorem 3.1 and Theorem 3.2, Theorem 3.3 are given now (see [Vo] for applications to inverse source problem). Recall that  $\omega$  is an open and nonempty subset of  $\Omega$ .

*Corollary 3.1 .- Let  $0 < L < T$  and  $\varepsilon > 0$ . If one of the statement of Theorem 3.1 holds then for any  $y_e \in L^2 (\Omega)$ , there is  $f \in L^2 (\omega)$  such that the solution  $y$  to*

$$\begin{cases} y' (t) + Py (t) = 0 , & t \in (0, T) \setminus \{L\} , \\ y (0) = y_e , \\ y (L) = y (L_-) + 1_\omega f , \end{cases}$$

*satisfies*

$$\|y (T)\|^2 \leq \varepsilon \|y_e\|^2 \text{ and } \|f\|_\omega^2 \leq e^{C_3 \left(1 + \left(\frac{1}{T-L}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{T-L} \ln(e + \frac{1}{\varepsilon})\right)^\sigma} \|y_e\|^2 .$$

Proof .- We apply Theorem 3.2 with  $\ell = p_\sigma(T - L, \varepsilon) = e^{C_3 \left(1 + (\frac{1}{T-L})^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{T-L} \ln(e + \frac{1}{\varepsilon})\right)^\sigma}$  given by Theorem 3.1 and  $T_0 = 0$ ,  $T_1 = L$ ,  $T_2 = T$  (knowing that  $\|u(T)\| \leq \|u(T - L)\|$ ).  $\square$

Corollary 3.2 .- Let  $0 < L < T$  and  $\varepsilon > 0$ . If one of the statement of Theorem 3.1 holds then for any  $y_d \in L^2(\Omega)$  such that  $\langle Py_d, y_d \rangle < +\infty$ , there is  $f \in L^2(\omega)$  such that the solution  $y$  to

$$\begin{cases} y'(t) + Py(t) = 0, & t \in (0, T) \setminus \{L\}, \\ y(0) = 0, \\ y(L) = y(L_-) + 1_\omega f, \end{cases}$$

satisfies  $\|y(T) - y_d\|^2 \leq \varepsilon \langle Py_d, y_d \rangle$  and

$$\|f\|_\omega^2 \leq \frac{1}{\lambda_1} e^{2C_4 \left(1 + (\frac{1}{T-L})^{\frac{\sigma}{1-\sigma}}\right)} e^{2 \left[ \frac{T}{\varepsilon} + \left(\frac{C_4}{T-L} \ln\left(\sqrt{\frac{1}{\varepsilon \lambda_1}} e^{\frac{2T}{\varepsilon}}\right)\right)^\sigma \right]} \langle Py_d, y_d \rangle.$$

Proof .- Recall that  $\frac{d}{dt} \langle P^{-1}u(t), u(t) \rangle + 2\|u(t)\|^2 = 0$  and it can be written as

$$\frac{1}{2} \frac{d}{dt} \langle P^{-1}u, u \rangle + N(t) \langle P^{-1}u, u \rangle = 0 \text{ with } N(t) = \frac{\|u(t)\|^2}{\langle P^{-1}u(t), u(t) \rangle}.$$

In the spirit of [BT] (see also [Ph]), one can check that  $N'(t) \leq 0$  by using Cauchy-Schwarz inequality:  $\|u\|^2 \leq \langle P^{-1}u, u \rangle \langle Pu, u \rangle$  and  $\frac{d}{dt} \|u\|^2 + 2 \langle Pu, u \rangle = 0$ . Therefore,

$$\langle P^{-1}u(0), u(0) \rangle \leq e^{2N(0)T} \langle P^{-1}u(T), u(T) \rangle \leq \frac{1}{\lambda_1} e^{2N(0)T} \|u(T)\|^2.$$

But by Theorem 3.1, it holds

$$\|u(T)\| \leq e^{C_4 \left(1 + (\frac{1}{T})^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_4}{T} \ln\left(\frac{\|u(0)\|}{\|u(T)\|}\right)\right)^\sigma} \|u(T)\|_\omega.$$

Therefore,

$$\langle P^{-1}u(0), u(0) \rangle \leq \frac{1}{\lambda_1} e^{2N(0)T} e^{2C_4 \left(1 + (\frac{1}{T})^{\frac{\sigma}{1-\sigma}}\right)} e^{2 \left(\frac{C_4}{T} \ln\left(\frac{\|u(0)\|}{\|u(T)\|}\right)\right)^\sigma} \|u(T)\|_\omega^2$$

which implies, using  $\frac{\|u(0)\|}{\|u(T)\|} \leq \frac{\|u(0)\|}{\sqrt{\lambda_1} \langle P^{-1}u(T), u(T) \rangle} \leq \sqrt{\frac{N(0)}{\lambda_1}} e^{2N(0)T}$ , the following estimate

$$\langle P^{-1}u(0), u(0) \rangle \leq \frac{1}{\lambda_1} e^{2N(0)T} e^{2C_4 \left(1 + (\frac{1}{T})^{\frac{\sigma}{1-\sigma}}\right)} e^{2 \left(\frac{C_4}{T} \ln\left(\sqrt{\frac{N(0)}{\lambda_1}} e^{2N(0)T}\right)\right)^\sigma} \|u(T)\|_\omega^2.$$

One conclude by distinguishing the case  $N(0) \leq 1/\varepsilon$  and the case  $N(0) > 1/\varepsilon$ , that for any  $\varepsilon, T > 0$ ,

$$\langle P^{-1}u(0), u(0) \rangle \leq \frac{1}{\lambda_1} e^{\frac{2T}{\varepsilon}} e^{2C_4 \left(1 + (\frac{1}{T})^{\frac{\sigma}{1-\sigma}}\right)} e^{2 \left(\frac{C_4}{T} \ln\left(\sqrt{\frac{1}{\varepsilon \lambda_1}} e^{\frac{2T}{\varepsilon}}\right)\right)^\sigma} \|u(T)\|_\omega^2 + \varepsilon \|u(0)\|^2.$$

It remains to apply Theorem 3.3 with  $\ell = \frac{1}{\lambda_1} e^{\frac{2T}{\varepsilon}} e^{2C_4 \left(1 + (\frac{1}{T-L})^{\frac{\sigma}{1-\sigma}}\right)} e^{2 \left(\frac{C_4}{T-L} \ln\left(\sqrt{\frac{1}{\varepsilon \lambda_1}} e^{\frac{2T}{\varepsilon}}\right)\right)^\sigma}$  and  $T_0 = 0$ ,  $T_1 = L$ ,  $T_2 = T$ .  $\square$

### 3.4 Null controllability with measurable set in time

Recall that  $\omega$  is an open and nonempty subset of  $\Omega$ .

Theorem 3.4 .- *Let  $T > 0$  and  $E \subset (0, T)$  a set of positive measure. If one of the statement of Theorem 3.1 holds then for any  $y^0 \in L^2(\Omega)$ , there is  $f \in L^2(\omega \times E)$  such that the solution  $y$  to*

$$\begin{cases} y'(t) + Py(t) = 1_{\omega \times E} f, & t \in (0, T), \\ y(0) = y^0, \end{cases}$$

satisfies  $y(T) = 0$ .

Proof .- The proof is divided into three steps.

*Step 1: Observability estimate with measurable set in time.* Based on a telescoping series method (see [Mi], [Mi2] and already exploited in [PW], [PWZ], [AEWZ], [EMZ], [Z], [WZ], [LiZ], [YZ], [Ph2]), the statement (ii) in Theorem 3.1 implies the following observability: The solution  $u$  to

$$\begin{cases} u'(t) + Pu(t) = 0, & t \in (0, T), \\ u(0) \in L^2(\Omega), \end{cases}$$

satisfies

$$\|u(T)\|^2 \leq K \int_E \|u(T-t)\|_\omega^2 dt.$$

Here,  $K$  is a constant only depending on  $(P, \Omega, \omega, \sigma, |E|)$ . Further, if  $E = (0, T)$ , then  $K = C \exp\left(\frac{C}{T^{1-\sigma}}\right)$  for some  $C = C(P, \Omega, \omega, \sigma)$ .

*Step 2: Approximate controllability.* Let  $\varepsilon > 0$ . Consider the functional  $J_\varepsilon$  defined on  $L^2(\Omega)$  given by

$$J_\varepsilon(u_0) = \frac{K}{2} \int_E \|u(T-t)\|_\omega^2 dt + \frac{\varepsilon}{2} \|u_0\|^2 - \langle y^0, u(T) \rangle,$$

where

$$\begin{cases} u'(t) + Pu(t) = 0, & t \in (0, T), \\ u(0) = u_0. \end{cases}$$

Notice that  $J_\varepsilon$  is strictly convex,  $C^1$  and coercive and therefore  $J_\varepsilon$  has a unique minimizer  $w_{\varepsilon,0} \in L^2(\Omega)$ , i.e.  $J_\varepsilon(w_{\varepsilon,0}) = \min_{u_0 \in L^2(\Omega)} J_\varepsilon(u_0)$ . Set

$$\begin{cases} w'_\varepsilon(t) + Pw_\varepsilon(t) = 0, & t \in (0, T), \\ w_\varepsilon(0) = w_{\varepsilon,0}, \end{cases} \quad \text{and} \quad \begin{cases} h'(t) + Ph(t) = 0, & t \in (0, T), \\ h(0) = h_0. \end{cases}$$

Since  $J'_\varepsilon(w_{\varepsilon,0})h_0 = 0$  for any  $h_0 \in L^2(\Omega)$ , we have

$$K \int_E \langle w_\varepsilon(T-t), h(T-t) \rangle_\omega dt + \varepsilon \langle w_{\varepsilon,0}, h_0 \rangle - \langle y^0, h(T) \rangle = 0 \quad \forall h_0 \in L^2(\Omega).$$

But the solution  $y_\varepsilon$  to

$$\begin{cases} y'_\varepsilon(t) + Py_\varepsilon(t) = 1_{\omega \times E} f_\varepsilon, & t \in (0, T), \\ y_\varepsilon(0) = y^0, \end{cases}$$



satisfies

$$\langle y_\varepsilon(T), u(0) \rangle - \langle y^0, u(T) \rangle = \int_E \langle f_\varepsilon(\cdot, t), u(T-t) \rangle_\omega dt \quad \forall u(0) \in L^2(\Omega)$$

which means

$$- \int_E \langle f_\varepsilon(\cdot, t), h(T-t) \rangle_\omega dt + \langle y_\varepsilon(T), h_0 \rangle - \langle y^0, h(T) \rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

By choosing  $f_\varepsilon(\cdot, t) = -Kw_\varepsilon(T-t)$ , we deduce that the solution  $y_\varepsilon$  satisfies

$$\varepsilon w_{\varepsilon,0} = y_\varepsilon(T) .$$

Further,

$$K \int_E \|w_\varepsilon(T-t)\|_\omega^2 dt + \varepsilon \|w_{\varepsilon,0}\|^2 = \frac{1}{K} \int_E \|f_\varepsilon(\cdot, t)\|_\omega^2 dt + \frac{1}{\varepsilon} \|y_\varepsilon(T)\|^2 .$$

Moreover, taking  $h_0 = w_{\varepsilon,0}$  into  $J'_\varepsilon(w_{\varepsilon,0})h_0 = 0$ , we get

$$K \int_E \|w_\varepsilon(T-t)\|_\omega^2 dt + \varepsilon \|w_{\varepsilon,0}\|^2 - \langle y^0, w_\varepsilon(T) \rangle = 0 .$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} K \int_E \|w_\varepsilon(T-t)\|_\omega^2 dt + \varepsilon \|w_{\varepsilon,0}\|^2 &\leq \|y^0\| \|w_\varepsilon(T)\| \\ &\leq \|y^0\| \left( K \int_E \|w_\varepsilon(T-t)\|_\omega^2 dt \right)^{1/2} \end{aligned}$$

where in the last line, we used the observability estimate with measurable set in time. Therefore, we get

$$K \int_E \|w_\varepsilon(T-t)\|_\omega^2 dt + 2\varepsilon \|w_{\varepsilon,0}\|^2 \leq \|y^0\|^2 ,$$

that is,

$$\frac{1}{K} \int_E \|f_\varepsilon(\cdot, t)\|_\omega^2 dt + \frac{2}{\varepsilon} \|y_\varepsilon(T)\|^2 \leq \|y^0\|^2$$

where

$$\begin{cases} y'_\varepsilon(t) + Py_\varepsilon(t) = 1_{\omega \times E} f_\varepsilon , & t \in (0, T) , \\ y_\varepsilon(0) = y^0 , \\ f_\varepsilon(x, t) = -Kw_\varepsilon(x, T-t) , & (x, t) \in \Omega \times (0, T) , \\ w'_\varepsilon(t) + Pw_\varepsilon(t) = 0 , & t \in (0, T) , \\ w_\varepsilon(T) = \frac{1}{\varepsilon} y_\varepsilon(T) . \end{cases}$$

*Step 3: Convergence of the control function.* We refer to [Zu, p.571]. Since  $w_\varepsilon(T-\cdot)$  is bounded in  $L^2(\omega \times E)$  and  $\sqrt{\varepsilon}w_{\varepsilon,0}$  is bounded in  $L^2(\Omega)$ , one can deduce that, for some function  $w(T-\cdot)$  in  $L^2(\omega \times E)$ ,  $w_\varepsilon(T-\cdot)$  weakly converge to  $w(T-\cdot)$  in  $L^2(\omega \times E)$  and  $\varepsilon w_{\varepsilon,0}$  tends to zero in  $L^2(\Omega)$ . Therefore the identity

$$K \int_E \langle w_\varepsilon(T-t), h(T-t) \rangle_\omega dt + \varepsilon \langle w_{\varepsilon,0}, h_0 \rangle - \langle y^0, h(T) \rangle = 0 \quad \forall h_0 \in L^2(\Omega) ,$$

becomes when  $\varepsilon \rightarrow 0$ , as

$$K \int_E \langle w(T-t), h(T-t) \rangle_\omega dt - \langle y^0, h(T) \rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

But the solution  $y$  to

$$\begin{cases} y'(t) + Py(t) = 1_{\omega \times E} f, & t \in (0, T) , \\ y(0) = y^0, \end{cases}$$

satisfies

$$- \int_E \langle f(\cdot, t), h(T-t) \rangle_\omega dt + \langle y(T), h_0 \rangle - \langle y^0, h(T) \rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

By choosing  $f(\cdot, t) = -Kw(T-t)$ , it follows that the solution  $y$  satisfies

$$y(T) = 0 .$$

This completes the proof. □

### 3.5 Finite time stabilization

Recall that  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  are the norm and the inner product of  $L^2(\Omega)$  respectively.

Assume that there are two positive constants  $c = c(\Omega)$  and  $\rho = \rho(d)$  such that

$$\text{Card} \{ \lambda_i \leq \Lambda \} = \sum_{\lambda_i \leq \Lambda} 1 \leq c\Lambda^{1/\rho} .$$

Such estimate can be provided by the Weyl asymptotic formula  $\lambda_k \sim C(\Omega) k^\rho$  as  $k \rightarrow \infty$ . In particular, if  $P = -\Delta$ , then  $\rho = \frac{2}{d}$ ; And if  $P = \Delta^2$ , then  $\rho = \frac{4}{d}$ . In the case of the one-dimensional degenerate operator  $P = \mathcal{P}$ , we have  $\rho = 2$ .

Define an increasing sequence  $(t_m)_{m \geq 0}$  converging to  $T > 0$  by

$$t_m = T \left( 1 - \frac{1}{b^m} \right) \text{ for some } b > 1 .$$

Introduce a linear bounded operator  $\mathcal{F}_m$  from  $L^2(\Omega)$  into  $L^2(\omega)$  in the following manner:

$$\begin{aligned} \mathcal{F}_m : L^2(\Omega) &\rightarrow L^2(\omega) \\ \vartheta &\mapsto \sum_{\lambda_j \leq \Lambda_m} \langle \vartheta, \Phi_j \rangle f_j \end{aligned}$$

where

$$\Lambda_m := \lambda_1 + \left( \frac{\eta}{T} \frac{b}{b-1} \right) b^{(\beta+1)m} \text{ with } \eta > 1, \quad \beta := \frac{\sigma}{1-\sigma} ,$$

and  $f_j$  is the impulse control of the heat equation associated to the eigenfunction  $\Phi_j$  (see Corollary 3.1):

$$\begin{cases} y'_j(t) + P y_j(t) = 0, & t \in (t_m, t_{m+1}) \setminus \left\{ \frac{t_m + t_{m+1}}{2} \right\}, \\ y_j(t_m) = \Phi_j, \\ y_j\left(\frac{t_m + t_{m+1}}{2}\right) = y_j\left(\left(\frac{t_m + t_{m+1}}{2}\right)_-\right) + 1_\omega f_j, \end{cases}$$

satisfying

$$\|y_j(t_{m+1})\|^2 \leq \frac{e^{-\eta b^{\beta m}}}{\sum_{\lambda_i \leq \Lambda_m} 1} \text{ and } \|f_j\|_\omega^2 \leq e^{C_3 \left(1 + \left(\frac{2}{t_{m+1} - t_m}\right)^\beta\right)} e^{\left(\frac{2C_3}{t_{m+1} - t_m} \ln\left(e + e^{\eta b^{\beta m}} \sum_{\lambda_i \leq \Lambda_m} 1\right)\right)^\sigma}.$$

Here,  $C_3 > 0$  and  $\sigma \in (0, 1)$  are the constants given in Theorem 3.1. Notice that

$$\|\mathcal{F}_m\|_{L^2(\Omega) \rightarrow L^2(\omega)}^2 \leq \sum_{\lambda_j \leq \Lambda_m} \|f_j\|_\omega^2.$$

Theorem 3.5 .- Let  $\omega$  be an open and nonempty subset of  $\Omega$ . Suppose that one of the statement of Theorem 3.1 holds and

$$\text{Card}\{\lambda_i \leq \Lambda\} \leq c\Lambda^{1/\rho} \quad \text{for any } \Lambda > 0.$$

Then, for any  $T > 0$  there are  $b, \eta > 1$  and  $C, K > 0$  such that for any  $z_0 \in L^2(\Omega)$ , the solution  $z$  to

$$\begin{cases} z'(t) + Pz(t) = 0, & t \in \mathbb{R}^+ \setminus \bigcup_{m \geq 0} \left(\frac{t_m + t_{m+1}}{2}\right), \\ z\left(\frac{t_m + t_{m+1}}{2}\right) = z\left(\left(\frac{t_m + t_{m+1}}{2}\right)_-\right) + 1_\omega \mathcal{F}_m(z(t_m)), \text{ for any integer } m \geq 0, \\ z(0) = z_0, \end{cases}$$

satisfies  $\|z(t)\| \leq C e^{-\frac{1}{K} \left(\frac{T}{T-t}\right)^{\frac{\sigma}{1-\sigma}}} \|z_0\|$  for any  $0 \leq t < T_-$ . Further,  $\lim_{m \rightarrow \infty} \|\mathcal{F}_m(z(t_m))\| = 0$ .

Proof .- We start to focus on the solution  $z$  on interval  $(t_m, t_{m+1})$  with initial data  $z(t_m) = \sum_{j \geq 1} a_j \Phi_j$  in  $L^2(\Omega)$ . Introduce the initial datum  $\phi(t_m) = \sum_{\lambda_j > \Lambda_m} a_j \Phi_j$  and  $\psi(t_m) = \sum_{\lambda_j \leq \Lambda_m} a_j \Phi_j$  associated to the solution of  $\phi'(t) + P\phi(t) = 0$  and

$$\begin{cases} \psi'(t) + P\psi(t) = 0, & t \in (t_m, t_{m+1}) \setminus \left\{ \frac{t_m + t_{m+1}}{2} \right\}, \\ \psi\left(\frac{t_m + t_{m+1}}{2}\right) = \psi\left(\left(\frac{t_m + t_{m+1}}{2}\right)_-\right) + 1_\omega \sum_{\lambda_j \leq \Lambda_m} a_j f_j. \end{cases}$$

Therefore,  $\phi(t_{m+1}) = \sum_{\lambda_j > \Lambda_m} a_j e^{-\lambda_j(t_{m+1} - t_m)} \Phi_j$  and

$$\|\phi(t_{m+1})\| \leq e^{-\Lambda_m(t_{m+1} - t_m)} \|z(t_m)\|.$$

But we have chosen  $\Lambda_m > \lambda_1$  in order that  $\eta b^{\beta m} \leq \Lambda_m (t_{m+1} - t_m)$ . It implies that

$$\|\phi(t_{m+1})\| \leq e^{-\eta b^{\beta m}} \|z(t_m)\| .$$

On the other hand, the solution  $\psi$  satisfies  $\psi = \sum_{\lambda_j \leq \Lambda_m} a_j y_j$  and

$$\|\psi(t_{m+1})\| \leq \sum_{\lambda_j \leq \Lambda_m} |a_j| \sqrt{\frac{e^{-\eta b^{\beta m}}}{\sum_{\lambda_i \leq \Lambda_m} 1}} \leq e^{-\frac{1}{2}\eta b^{\beta m}} \|v(t_m)\| .$$

Consequently, we have

$$\|z(t_{m+1})\| \leq \|\phi(t_{m+1})\| + \|\psi(t_{m+1})\| \leq e^{1-\frac{1}{2}\eta b^{\beta m}} \|z(t_m)\| ,$$

which implies by induction that for any  $m \geq 1$ ,

$$\|z(t_m)\|^2 \leq e^{2m-\eta b^{\beta m}} \|z(t_0)\|^2 .$$

Now, we treat the boundeness of the control associated to  $\psi$ : Notice that  $\sum_{\lambda_j \leq \Lambda_m} a_j f_j := \mathcal{F}_m(z(t_m))$ , and then, by Cauchy-Schwarz and Young inequalities,

$$\begin{aligned} \|\mathcal{F}_m(z(t_m))\|_{\omega}^2 &\leq \int_{\omega} \left( \sum_{\lambda_j \leq \Lambda_m} |a_j| |f_j| \right)^2 \leq \sum_{\lambda_j \leq \Lambda_m} |a_j|^2 \sum_{j \leq \Lambda_m} \|f_j\|_{\omega}^2 \\ &\leq \|z(t_m)\|^2 \sum_{\lambda_j \leq \Lambda_m} e^{C_3 \left(1 + \left(\frac{2}{t_{m+1}-t_m}\right)^{\beta}\right)} e^{\left(\frac{2C_3}{t_{m+1}-t_m} \ln \left(e + \sum_{\lambda_i \leq \Lambda_m} 1 e^{\eta b^{\beta m}}\right)\right)^{\sigma}} \\ &\leq \|z(t_m)\|^2 e^{C_3 \left(1 + \left(\frac{2}{t_{m+1}-t_m}\right)^{\beta}\right)} \sum_{\lambda_j \leq \Lambda_m} e^{\left(\frac{4C_3}{t_{m+1}-t_m}\right)^{\beta} + \frac{1}{2} \ln \left(\sum_{\lambda_i \leq \Lambda_m} 1 e^{\eta b^{\beta m}}\right)} \\ &\leq \|z(t_m)\|^2 e^{C_3 \left(1 + \left(\frac{2}{t_{m+1}-t_m}\right)^{\beta}\right)} e^{\left(\frac{4C_3}{t_{m+1}-t_m}\right)^{\frac{\sigma}{1-\sigma}}} e^{\frac{1}{2}\eta b^{\beta m}} \left( \sum_{\lambda_i \leq \Lambda_m} 1 \right)^{\theta+1} \\ &\leq e^{2m-\eta b^{\beta m}} \|z(t_0)\|^2 e^{C_3 \left(1 + \left(\frac{2}{T} \frac{b^{m+1}}{b-1}\right)^{\beta}\right)} e^{\left(\frac{4C_3}{T} \frac{b^{m+1}}{b-1}\right)^{\beta}} e^{\frac{1}{2}\eta b^{\beta m}} \left(c \Lambda_m^{1/\rho}\right)^{\theta+1} \\ &\leq e^{2m-\frac{1}{2}\eta b^{\beta m}} \|z(t_0)\|^2 e^{C_3} e^{(C_3+(2C_3)^{\beta}) \left(\frac{2}{T} \frac{b}{b-1}\right)^{\beta} b^{\beta m}} \left(c \left(\lambda_1 + \left(\frac{\eta}{T} \frac{b}{b-1}\right) b^{(\beta+1)m}\right)\right)^{\frac{\theta+1}{\rho}} , \end{aligned}$$

where in the last line we used the definition of  $\Lambda_m$ . Next, we choose  $\eta > 1$ , precisely

$$\eta = 1 + 4 \left(C_3 + (2C_3)^{\beta}\right) \left(\frac{2}{T} \frac{b}{b-1}\right)^{\beta}$$

in order that  $-\frac{1}{2}\eta b^{\beta m} + \left(C_3 + \left(\frac{C_3}{\theta}\right)^{\beta}\right) \left(\frac{2}{T} \frac{b}{b-1}\right)^{\beta} b^{\beta m} \leq -\frac{1}{4}\eta b^{\beta m}$ .

Since  $b > 1$  and  $b^{(\beta+1)(\theta+1)m/\rho} \leq \left(\frac{8(\beta+1)(\theta+1)}{\beta\rho\eta}\right)^{\frac{(\beta+1)(\theta+1)}{\beta\rho}} e^{\frac{1}{8}\eta b^{\beta m}}$ , we obtain for some constant  $C_5 := \left(c \left(\lambda_1 + \left(\frac{\eta}{T} \frac{b}{b-1}\right)\right)\right)^{\frac{\theta+1}{\rho}} > 0$  that for any  $m \geq 1$ ,

$$\|\mathcal{F}_m(v(t_m))\|_{\omega}^2 \leq C_5 e^{2m-\frac{1}{8}\eta b^{\beta m}} \|v(0)\|^2 .$$

Finally, let  $t \geq 0$ , then there is  $m \geq 0$  such that  $t \in [t_m, t_{m+1}]$ . We distinguish four cases: If  $t \in [0, t_1/2)$ , then

$$\|z(t)\|^2 \leq \|z(0)\|^2 ;$$

If  $t \in [t_1/2, t_1)$ , then

$$\|z(t)\|^2 \leq \|z((t_1/2)_-) + 1_\omega \mathcal{F}_0(z(t_0))\|^2 \leq 2(1 + \|\mathcal{F}_0\|^2) \|z(0)\|^2 .$$

If  $t \in [t_m, \frac{t_m+t_{m+1}}{2})$  and  $m \geq 1$ , then

$$\|z(t)\|^2 \leq \|z(t_m)\|^2 \leq e^{2m-\eta b^{\beta m}} \|z(0)\|^2 ;$$

If  $t \in [\frac{t_m+t_{m+1}}{2}, t_{m+1})$  and  $m \geq 1$ , then

$$\begin{aligned} \|z(t)\|^2 &\leq \left\| v\left(\left(\frac{t_m+t_{m+1}}{2}\right)_-\right) + 1_\omega \mathcal{F}_m(z(t_m)) \right\|^2 \\ &\leq 2(1 + C_5) b^{(\beta+1)(\theta+1)m/\rho} e^{2m-\frac{1}{8}\eta b^{\beta m}} \|z(0)\|^2 . \end{aligned}$$

Consequently, for any  $t \in [t_m, t_{m+1}]$ , it holds

$$b^{\beta m} \leq \left(\frac{T}{T-t}\right)^\beta \leq b^{\beta m+\beta}$$

and

$$\|z(t)\|^2 \leq 2(1 + C_5 + \|\mathcal{F}_0\|^2) e^{-\frac{1}{16}\eta b^{\beta m}} \|z(0)\|^2$$

by choosing  $b = e^{32/(\beta\eta)}$ . One conclude that  $e^{-\frac{1}{16}\eta b^{\beta m}} \leq e^{-\frac{1}{16}\eta(\frac{T}{b} \frac{1}{T-t})^\beta}$  and

$$\|z(t)\|^2 \leq 2(1 + C_5 + \|\mathcal{F}_0\|^2) e^{-\frac{1}{16}\eta e^{32/\eta}(\frac{T}{T-t})^\beta} \|z(0)\|^2 .$$

This completes the proof. □

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